In addition to *Functional Analysis*, Second Edition, Walter Rudin is the author of two other books: *Principles of Mathematical Analysis* and *Real and Complex Analysis*, whose widespread use is illustrated by the fact that they have been translated into a total of 13 languages. He wrote *Principles of Mathematical Analysis* while he was a C.L.E. Moore Instructor at the Massachusetts Institute of Technology—just two years after receiving his Ph.D. at Duke University. Later, he taught at the University of Rochester, and is now a Vilas Research Professor at the University of Wisconsin–Madison. In the past, he has spent leaves at Yale University, the University of California in La Jolla, and the University of Hawaii.

Dr. Rudin’s research has dealt mainly with harmonic analysis and with complex variables. He has written three research monographs on these topics: *Fourier Analysis on Groups*, *Function Theory in Polydiscs*, and *Function Theory in the Unit Ball of $C^n$*. 
CONTENTS

Preface

Part I General Theory

1 Topological Vector Spaces
   Introduction 3
   Separation properties 10
   Linear mappings 14
   Finite-dimensional spaces 16
   Metrization 18
   Boundedness and continuity 23
   Seminorms and local convexity 25
   Quotient spaces 30
   Examples 33
   Exercises 38

2 Completeness
   Baire category 42
   The Banach-Steinhaus theorem 43
   The open mapping theorem 47
   The closed graph theorem 50
   Bilinear mappings 52
   Exercises 53

3 Convexity
   The Hahn-Banach theorems 56
   Weak topologies 62
   Compact convex sets 68
   Vector-valued integration 77
   Holomorphic functions 82
   Exercises 85
## CONTENTS

### Part II Distributions and Fourier Transforms

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Test Functions and Distributions</td>
<td>149</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>149</td>
</tr>
<tr>
<td></td>
<td>Test function spaces</td>
<td>151</td>
</tr>
<tr>
<td></td>
<td>Calculus with distributions</td>
<td>157</td>
</tr>
<tr>
<td></td>
<td>Localization</td>
<td>162</td>
</tr>
<tr>
<td></td>
<td>Supports of distributions</td>
<td>164</td>
</tr>
<tr>
<td></td>
<td>Distributions as derivatives</td>
<td>167</td>
</tr>
<tr>
<td></td>
<td>Convolutions</td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>177</td>
</tr>
<tr>
<td>7</td>
<td>Fourier Transforms</td>
<td>182</td>
</tr>
<tr>
<td></td>
<td>Basic properties</td>
<td>182</td>
</tr>
<tr>
<td></td>
<td>Tempered distributions</td>
<td>189</td>
</tr>
<tr>
<td></td>
<td>Paley-Wiener theorems</td>
<td>196</td>
</tr>
<tr>
<td></td>
<td>Sobolev’s lemma</td>
<td>202</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>204</td>
</tr>
<tr>
<td>8</td>
<td>Applications to Differential Equations</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>Fundamental solutions</td>
<td>210</td>
</tr>
<tr>
<td></td>
<td>Elliptic equations</td>
<td>215</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>222</td>
</tr>
</tbody>
</table>
9 Tauberian Theory 226
Wiener's theorem 226
The prime number theorem 230
The renewal equation 236
Exercises 239

Part III Banach Algebras and Spectral Theory

10 Banach Algebras 245
Introduction 245
Complex homomorphisms 249
Basic properties of spectra 252
Symbolic calculus 258
The group of invertible elements 267
Lomonosov's invariant subspace theorem 269
Exercises 271

11 Commutative Banach Algebras 275
Ideals and homomorphisms 275
Gelfand transforms 280
Involutions 287
Applications to noncommutative algebras 292
Positive functionals 296
Exercises 301

12 Bounded Operators on a Hilbert Space 306
Basic facts 306
Bounded operators 309
A commutativity theorem 315
Resolutions of the identity 316
The spectral theorem 321
Eigenvalues of normal operators 327
Positive operators and square roots 330
The group of invertible operators 333
A characterization of $B^*$-algebras 336
An ergodic theorem 339
Exercises 341

13 Unbounded Operators 347
Introduction 347
Graphs and symmetric operators 351
The Cayley transform 356
Resolutions of the identity 360
The spectral theorem 368
Semigroups of operators 375
Exercises 385
Appendix A  Compactness and Continuity  391
Appendix B  Notes and Comments  397
Bibliography  412
List of Special Symbols  414
Index  417
Functional analysis is the study of certain topological-algebraic structures and of the methods by which knowledge of these structures can be applied to analytic problems.

A good introductory text on this subject should include a presentation of its axiomatics (i.e., of the general theory of topological vector spaces), it should treat at least a few topics in some depth, and it should contain some interesting applications to other branches of mathematics. I hope that the present book meets these criteria.

The subject is huge and is growing rapidly. (The bibliography in volume I of [4] contains 96 pages and goes only to 1957.) In order to write a book of moderate size, it was therefore necessary to select certain areas and to ignore others. I fully realize that almost any expert who looks at the table of contents will find that some of his or her (and my) favorite topics are missing, but this seems unavoidable. It was not my intention to write an encyclopedic treatise. I wanted to write a book that would open the way to further exploration.

This is the reason for omitting many of the more esoteric topics that might have been included in the presentation of the general theory of topological vector spaces. For instance, there is no discussion of uniform spaces, of Moore-Smith convergence, of nets, or of filters. The notion of completeness occurs only in the context of metric spaces. Bornological spaces are not mentioned, nor are barreled ones. Duality is of course presented, but not in its utmost generality. Integration of vector-valued functions is treated strictly as a tool; attention is confined to continuous integrands, with values in a Fréchet space.

Nevertheless, the material of Part I is fully adequate for almost all applications to concrete problems. And this is what ought to be stressed in such a course: The close interplay between the abstract and the concrete is
not only the most useful aspect of the whole subject but also the most fascinating one.

Here are some further features of the selected material. A fairly large part of the general theory is presented without the assumption of local convexity. The basic properties of compact operators are derived from the duality theory in Banach spaces. The Krein-Milman theorem on the existence of extreme points is used in several ways in Chapter 5. The theory of distributions and Fourier transforms is worked out in fair detail and is applied (in two very brief chapters) to two problems in partial differential equations, as well as to Wiener's tauberian theorem and two of its applications. The spectral theorem is derived from the theory of Banach algebras (specifically, from the Gelfand-Naimark characterization of commutative $B^*$-algebras); this is perhaps not the shortest way, but it is an easy one. The symbolic calculus in Banach algebras is discussed in considerable detail; so are involutions and positive functionals.

I assume familiarity with the theory of measure and Lebesgue integration (including such facts as the completeness of the $L^p$-spaces), with some basic properties of holomorphic functions (such as the general form of Cauchy's theorem, and Runge's theorem), and with the elementary topological background that goes with these two analytic topics. Some other topological facts are briefly presented in Appendix A. Almost no algebraic background is needed, beyond the knowledge of what a homomorphism is.

Historical references are gathered in Appendix B. Some of these refer to the original sources, and some to more recent books, papers, or expository articles in which further references can be found. There are, of course, many items that are not documented at all. In no case does the absence of a specific reference imply any claim to originality on my part.

Most of the applications are in Chapters 5, 8, and 9. Some are in Chapter 11 and in the more than 250 exercises; many of these are supplied with hints. The interdependence of the chapters is indicated in the diagram on the following page.

Most of the applications contained in Chapter 5 can be taken up well before the first four chapters are completed. It has therefore been suggested that it might be good pedagogy to insert them into the text earlier, as soon as the required theoretical background is established. However, in order not to interrupt the presentation of the theory in this way, I have instead started Chapter 5 with a short indication of the background that is needed for each item. This should make it easy to study the applications as early as possible, if so desired.

In the first edition, a fairly large part of Chapter 10 dealt with differentiation in Banach algebras. Twenty years ago this (then recent) material looked interesting and promising, but it does not seem to have led anywhere, and I have deleted it. On the other hand, I have added a few items which were easy to fit into the existing text: the mean ergodic theorem of
von Neumann, the Hille-Yosida theorem on semigroups of operators, a couple of fixed point theorems, Bonsall's surprising application of the closed range theorem, and Lomonosov's spectacular invariant subspace theorem. I have also rewritten a few sections in order to clarify certain details, and I have shortened and simplified some proofs.

Most of these changes have been made in response to much-appreciated suggestions by numerous friends and colleagues. I especially want to mention Justin Peters and Ralph Raimi, who wrote detailed critiques of the first edition, and the Russian translator of the first edition who added quite a few relevant footnotes to the text. My thanks to all of them!

Walter Rudin
PART I

GENERAL THEORY
Introduction

1.1 Many problems that analysts study are not primarily concerned with a single object such as a function, a measure, or an operator, but they deal instead with large classes of such objects. Most of the interesting classes that occur in this way turn out to be vector spaces, either with real scalars or with complex ones. Since limit processes play a role in every analytic problem (explicitly or implicitly), it should be no surprise that these vector spaces are supplied with metrics, or at least with topologies, that bear some natural relation to the objects of which the spaces are made up. The simplest and most important way of doing this is to introduce a norm. The resulting structure (defined below) is called a normed vector space, or a normed linear space, or simply a normed space.

Throughout this book, the term vector space will refer to a vector space over the complex field \( \mathbb{C} \) or over the real field \( \mathbb{R} \). For the sake of completeness, detailed definitions are given in Section 1.4.

1.2 Normed spaces A vector space \( X \) is said to be a normed space if to every \( x \in X \) there is associated a nonnegative real number \( \|x\| \), called the norm of \( x \), in such a way that
(a) \[ \|x + y\| \leq \|x\| + \|y\| \] for all \( x \) and \( y \) in \( X \),
(b) \[ \|\alpha x\| = |\alpha| \|x\| \] if \( x \in X \) and \( \alpha \) is a scalar,
(c) \( \|x\| > 0 \) if \( x \neq 0 \).

The word “norm” is also used to denote the function that maps \( x \) to \( \|x\| \).

Every normed space may be regarded as a metric space, in which the distance \( d(x, y) \) between \( x \) and \( y \) is \( \|x - y\| \). The relevant properties of \( d \) are

(i) \( 0 < d(x, y) < \infty \) for all \( x \) and \( y \),
(ii) \( d(x, y) = 0 \) if and only if \( x = y \),
(iii) \( d(x, y) = d(y, x) \) for all \( x \) and \( y \),
(iv) \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \).

In any metric space, the open ball with center at \( x \) and radius \( r \) is the set
\[ B_r(x) = \{ y : d(x, y) < r \} \]

In particular, if \( X \) is a normed space, the sets
\[ B_1(0) = \{ x : \|x\| < 1 \} \quad \text{and} \quad \bar{B}_1(0) = \{ x : \|x\| \leq 1 \} \]
are the open unit ball and the closed unit ball of \( X \), respectively.

By declaring a subset of a metric space to be open if and only if it is a (possibly empty) union of open balls, a topology is obtained. (See Section 1.5.) It is quite easy to verify that the vector space operations (addition and scalar multiplication) are continuous in this topology, if the metric is derived from a norm, as above.

A Banach space is a normed space which is complete in the metric defined by its norm; this means that every Cauchy sequence is required to converge.

1.3 Many of the best-known function spaces are Banach spaces. Let us mention just a few types: spaces of continuous functions on compact spaces; the familiar \( L^p \)-spaces that occur in integration theory; Hilbert spaces — the closest relatives of euclidean spaces; certain spaces of differentiable functions; spaces of continuous linear mappings from one Banach space into another; Banach algebras. All of these will occur later on in the text.

But there are also many important spaces that do not fit into this framework. Here are some examples:

(a) \( C(\Omega) \), the space of all continuous complex functions on some open set \( \Omega \) in a euclidean space \( R^n \).
(b) $H(\Omega)$, the space of all holomorphic functions in some open set $\Omega$ in the complex plane.

(c) $C^\infty_K$, the space of all infinitely differentiable complex functions on $\mathbb{R}^n$ that vanish outside some fixed compact set $K$ with nonempty interior.

(d) The test function spaces used in the theory of distributions, and the distributions themselves.

These spaces carry natural topologies that cannot be induced by norms, as we shall see later. They, as well as the normed spaces, are examples of topological vector spaces, a concept that pervades all of functional analysis.

After this brief attempt at motivation, here are the detailed definitions, followed (in Section 1.9) by a preview of some of the results of Chapter 1.

### 1.4 Vector spaces

The letters $\mathbb{R}$ and $\mathbb{C}$ will always denote the field of real numbers and the field of complex numbers, respectively. For the moment, let $\Phi$ stand for either $\mathbb{R}$ or $\mathbb{C}$. A scalar is a member of the scalar field $\Phi$. A vector space over $\Phi$ is a set $X$, whose elements are called vectors, and in which two operations, addition and scalar multiplication, are defined, with the following familiar algebraic properties:

(a) To every pair of vectors $x$ and $y$ corresponds a vector $x + y$, in such a way that

$$x + y = y + x \quad \text{and} \quad x + (y + z) = (x + y) + z;$$

$X$ contains a unique vector $0$ (the zero vector or origin of $X$) such that $x + 0 = x$ for every $x \in X$; and to each $x \in X$ corresponds a unique vector $-x$ such that $x + (-x) = 0$.

(b) To every pair $(\alpha, x)$ with $\alpha \in \Phi$ and $x \in X$ corresponds a vector $\alpha x$, in such a way that

$$1x = x, \quad \alpha(\beta x) = (\alpha \beta)x,$$

and such that the two distributive laws

$$\alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x$$

hold.

The symbol $0$ will of course also be used for the zero element of the scalar field.

A real vector space is one for which $\Phi = \mathbb{R}$; a complex vector space is one for which $\Phi = \mathbb{C}$. Any statement about vector spaces in which the scalar field is not explicitly mentioned is to be understood to apply to both of these cases.
If $X$ is a vector space, $A \subseteq X$, $B \subseteq X$, $x \in X$, and $\lambda \in \Phi$, the following notations will be used:

\[
x + A = \{ x + a : a \in A \},
\]

\[
x - A = \{ x - a : a \in A \},
\]

\[
A + B = \{ a + b : a \in A, b \in B \},
\]

\[
\lambda A = \{ \lambda a : a \in A \}.
\]

In particular (taking $\lambda = -1$), $-A$ denotes the set of all additive inverses of members of $A$.

A word of warning: With these conventions, it may happen that $2A \neq A + A$ (Exercise 1).

A set $Y \subseteq X$ is called a subspace of $X$ if $Y$ is itself a vector space (with respect to the same operations, of course). One checks easily that this happens if and only if $0 \in Y$ and

\[
\alpha Y + \beta Y \subseteq Y
\]

for all scalars $\alpha$ and $\beta$.

A set $C \subseteq X$ is said to be convex if

\[
tC + (1 - t)C \subseteq C \quad (0 \leq t \leq 1).
\]

In other words, it is required that $C$ should contain $tx + (1 - t)y$ if $x \in C$, $y \in C$, and $0 \leq t \leq 1$.

A set $B \subseteq X$ is said to be balanced if $\alpha B \subseteq B$ for every $\alpha \in \Phi$ with $|\alpha| \leq 1$.

A vector space $X$ has dimension $n$ ($\text{dim } X = n$) if $X$ has a basis $\{ u_1, \ldots, u_n \}$. This means that every $x \in X$ has a unique representation of the form

\[
x = \alpha_1 u_1 + \cdots + \alpha_n u_n \quad (\alpha_i \in \Phi).
\]

If $\text{dim } X = n$ for some $n$, $X$ is said to have finite dimension. If $X = \{0\}$, then $\text{dim } X = 0$.

Example. If $X = \mathcal{C}$ (a one-dimensional vector space over the scalar field $\mathcal{C}$), the balanced sets are $\mathcal{C}$, the empty set $\emptyset$, and every circular disc (open or closed) centered at 0. If $X = \mathbb{R}^2$ (a two-dimensional vector space over the scalar field $\mathbb{R}$), there are many more balanced sets; any line segment with midpoint at $(0, 0)$ will do. The point is that, in spite of the well-known and obvious identification of $\mathcal{C}$ with $\mathbb{R}^2$, these two are entirely different as far as their vector space structure is concerned.

1.5 Topological spaces A topological space is a set $S$ in which a collection $\tau$ of subsets (called open sets) has been specified, with the following
properties: \(S\) is open, \(\emptyset\) is open, the intersection of any two open sets is open, and the union of every collection of open sets is open. Such a collection \(\tau\) is called a topology on \(S\). When clarity seems to demand it, the topological space corresponding to the topology \(\tau\) will be written \((S, \tau)\) rather than \(S\).

Here is some of the standard vocabulary that will be used, if \(S\) and \(\tau\) are as above.

A set \(E \subset S\) is closed if and only if its complement is open. The closure \(\bar{E}\) of \(E\) is the intersection of all closed sets that contain \(E\). The interior \(E^\circ\) of \(E\) is the union of all open sets that are subsets of \(E\). A neighborhood of a point \(p \in S\) is any open set that contains \(p\). \((S, \tau)\) is a Hausdorff space, and \(\tau\) is a Hausdorff topology, if distinct points of \(S\) have disjoint neighborhoods. A set \(K \subset S\) is compact if every open cover of \(K\) has a finite subcover. A collection \(\tau' \subset \tau\) is a base for \(\tau\) if every member of \(\tau\) (that is, every open set) is a union of members of \(\tau'\). A collection \(\gamma\) of neighborhoods of a point \(p \in S\) is a local base at \(p\) if every neighborhood of \(p\) contains a member of \(\gamma\).

If \(E \subset S\) and if \(\sigma\) is the collection of all intersections \(E \cap V\), with \(V \in \tau\), then \(\sigma\) is a topology on \(E\), as is easily verified; we call this the topology that \(E\) inherits from \(S\).

If a topology \(\tau\) is induced by a metric \(d\) (see Section 1.2) we say that \(d\) and \(\tau\) are compatible with each other.

A sequence \(\{x_n\}\) in a Hausdorff space \(X\) converges to a point \(x \in X\) (or \(\lim_{n \to \infty} x_n = x\)) if every neighborhood of \(x\) contains all but finitely many of the points \(x_n\).

### 1.6 Topological vector spaces

Suppose \(\tau\) is a topology on a vector space \(X\) such that

(a) every point of \(X\) is a closed set, and

(b) the vector space operations are continuous with respect to \(\tau\).

Under these conditions, \(\tau\) is said to be a vector topology on \(X\), and \(X\) is a topological vector space.

Here is a more precise way of stating (a): For every \(x \in X\), the set \(\{x\}\) which has \(x\) as its only member is a closed set.

In many texts, (a) is omitted from the definition of a topological vector space. Since (a) is satisfied in almost every application, and since most theorems of interest require (a) in their hypotheses, it seems best to include it in the axioms. [Theorem 1.12 will show that (a) and (b) together imply that \(\tau\) is a Hausdorff topology.]

To say that addition is continuous means, by definition, that the mapping

\[
(x, y) \rightarrow x + y
\]
of the cartesian product $X \times X$ into $X$ is continuous: If $x_i \in X$ for $i = 1, 2$, and if $V$ is a neighborhood of $x_1 + x_2$, there should exist neighborhoods $V_i$ of $x_i$ such that

$$V_1 + V_2 \subseteq V.$$ 

Similarly, the assumption that scalar multiplication is continuous means that the mapping

$$(\alpha, x) \mapsto \alpha x$$

of $\Phi \times X$ into $X$ is continuous: If $x \in X$, $\alpha$ is a scalar, and $V$ is a neighborhood of $\alpha x$, then for some $r > 0$ and some neighborhood $W$ of $x$ we have $\beta W \subseteq V$ whenever $|\beta - \alpha| < r$.

A subset $E$ of a topological vector space is said to be bounded if to every neighborhood $V$ of $0$ in $X$ corresponds a number $s > 0$ such that $E \subseteq tV$ for every $t > s$.

1.7 Invariance Let $X$ be a topological vector space. Associate to each $a \in X$ and to each scalar $\lambda \neq 0$ the translation operator $T_a$ and the multiplication operator $M_\lambda$, by the formulas

$$T_a(x) = a + x, \quad M_\lambda(x) = \lambda x \quad (x \in X).$$

The following simple proposition is very important:

**Proposition.** $T_a$ and $M_\lambda$ are homeomorphisms of $X$ onto $X$.

**Proof.** The vector space axioms alone imply that $T_a$ and $M_\lambda$ are one-to-one, that they map $X$ onto $X$, and that their inverses are $T_{-a}$ and $M_{1/\lambda}$, respectively. The assumed continuity of the vector space operations implies that these four mappings are continuous. Hence each of them is a homeomorphism (a continuous mapping whose inverse is also continuous).

One consequence of this proposition is that every vector topology $\tau$ is translation-invariant (or simply invariant, for brevity): A set $E \subseteq X$ is open if and only if each of its translates $a + E$ is open. Thus $\tau$ is completely determined by any local base.

In the vector space context, the term local base will always mean a local base at $0$. A local base of a topological vector space $X$ is thus a collection $\mathcal{B}$ of neighborhoods of $0$ such that every neighborhood of $0$ contains a member of $\mathcal{B}$. The open sets of $X$ are then precisely those that are unions of translates of members of $\mathcal{B}$. 
A metric $d$ on a vector space $X$ will be called \textit{invariant} if
\[ d(x + z, y + z) = d(x, y) \]
for all $x, y, z$ in $X$.

1.8 \textbf{Types of topological vector spaces} \hspace{1em} In the following definitions, $X$ always denotes a topological vector space, with topology $\tau$.

(a) $X$ is \textit{locally convex} if there is a local base $\mathscr{B}$ whose members are convex.

(b) $X$ is \textit{locally bounded} if $0$ has a bounded neighborhood.

(c) $X$ is \textit{locally compact} if $0$ has a neighborhood whose closure is compact.

(d) $X$ is \textit{metrizable} if $\tau$ is compatible with some metric $d$.

(e) $X$ is an $F$-space if its topology $\tau$ is induced by a complete invariant metric $d$. (Compare Section 1.25.)

(f) $X$ is a \textit{Fréchet space} if $X$ is a locally convex $F$-space.

(g) $X$ is \textit{normable} if a norm exists on $X$ such that the metric induced by the norm is compatible with $\tau$.

(h) \textit{Normed spaces} and \textit{Banach spaces} have already been defined (Section 1.2).

(i) $X$ has the \textit{Heine-Borel property} if every closed and bounded subset of $X$ is compact.

The terminology of (e) and (f) is not universally agreed upon: In some texts, local convexity is omitted from the definition of a Fréchet space, whereas others use $F$-space to describe what we have called Fréchet space.

1.9 \hspace{1em} Here is a list of some relations between these properties of a topological vector space $X$.

(a) If $X$ is locally bounded, then $X$ has a countable local base [part (c) of Theorem 1.15].

(b) $X$ is metrizable if and only if $X$ has a countable local base (Theorem 1.24).

(c) $X$ is normable if and only if $X$ is locally convex and locally bounded (Theorem 1.39).

(d) $X$ has finite dimension if and only if $X$ is locally compact (Theorems 1.21, 1.22).

(e) If a locally bounded space $X$ has the Heine-Borel property, then $X$ has finite dimension (Theorem 1.23).
The spaces $H(\Omega)$ and $C_c^\infty$ mentioned in Section 1.3 are infinite-dimensional Fréchet spaces with the Heine-Borel property (Sections 1.45, 1.46). They are therefore not locally bounded, hence not normable; they also show that the converse of (a) is false.

On the other hand, there exist locally bounded $F$-spaces that are not locally convex (Section 1.47).

**Separation Properties**

1.10 **Theorem** Suppose $K$ and $C$ are subsets of a topological vector space $X$, $K$ is compact, $C$ is closed, and $K \cap C = \emptyset$. Then $0$ has a neighborhood $V$ such that

$$(K + V) \cap (C + V) = \emptyset.$$ 

Note that $K + V$ is a union of translates $x + V$ of $V$ ($x \in K$). Thus $K + V$ is an open set that contains $K$. The theorem thus implies the existence of disjoint open sets that contain $K$ and $C$, respectively.

**Proof.** We begin with the following proposition, which will be useful in other contexts as well:

*If $W$ is a neighborhood of $0$ in $X$, then there is a neighborhood $U$ of $0$ which is symmetric (in the sense that $U = -U$) and which satisfies $U + U \subseteq W$.***

To see this, note that $0 + 0 = 0$, that addition is continuous, and that $0$ therefore has neighborhoods $V_1, V_2$ such that $V_1 + V_2 \subseteq W$. If

$$U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2),$$

then $U$ has the required properties.

The proposition can now be applied to $U$ in place of $W$ and yields a new symmetric neighborhood $U$ of $0$ such that

$$U + U + U + U \subseteq W.$$ 

It is clear how this can be continued.

If $K = \emptyset$, then $K + V = \emptyset$, and the conclusion of the theorem is obvious. We therefore assume that $K \neq \emptyset$, and consider a point $x \in K$. Since $C$ is closed, since $x$ is not in $C$, and since the topology of $X$ is invariant under translations, the preceding proposition shows that $0$ has a symmetric neighborhood $V_x$ such that $x + V_x + V_x + V_x$ does not intersect $C$; the symmetry of $V_x$ shows then that

$$(x + V_x + V_x) \cap (C + V_x) = \emptyset.$$ (1)
Since $K$ is compact, there are finitely many points $x_1, \ldots, x_n$ in $K$ such that
\[ K \subseteq (x_1 + V_{x_1}) \cup \cdots \cup (x_n + V_{x_n}). \]
Put $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then
\[ K + V \subseteq \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \subseteq \bigcup_{i=1}^{n} (x_i + V_{x_i} + V_{x_i}), \]
and no term in this last union intersects $C + V$, by (1). This completes the proof.

Since $C + V$ is open, it is even true that the closure of $K + V$ does not intersect $C + V$; in particular, the closure of $K + V$ does not intersect $C$. The following special case of this, obtained by taking $K = \{0\}$, is of considerable interest.

1.11 Theorem If $\mathcal{B}$ is a local base for a topological vector space $X$, then every member of $\mathcal{B}$ contains the closure of some member of $\mathcal{B}$.

So far we have not used the assumption that every point of $X$ is a closed set. We now use it and apply Theorem 1.10 to a pair of distinct points in place of $K$ and $C$. The conclusion is that these points have disjoint neighborhoods. In other words, the Hausdorff separation axiom holds:

1.12 Theorem Every topological vector space is a Hausdorff space.

We now derive some simple properties of closures and interiors in a topological vector space. See Section 1.5 for the notations $\bar{E}$ and $E^\circ$. Observe that a point $p$ belongs to $\bar{E}$ if and only if every neighborhood of $p$ intersects $E$.

1.13 Theorem Let $X$ be a topological vector space.

(a) If $A \subset X$ then $\bar{A} = \bigcap (A + V)$, where $V$ runs through all neighborhoods of 0.
(b) If $A \subset X$ and $B \subset X$, then $\bar{A} + \bar{B} \subseteq \overline{A + B}$.
(c) If $Y$ is a subspace of $X$, so is $\bar{Y}$.
(d) If $C$ is a convex subset of $X$, so are $\bar{C}$ and $C^\circ$.
(e) If $B$ is a balanced subset of $X$, so is $\bar{B}$; if also $0 \in B^\circ$ then $B^\circ$ is balanced.
(f) If $E$ is a bounded subset of $X$, so is $\bar{E}$. 
PROOF. (a) $x \in \widetilde{A}$ if and only if $(x + V) \cap A \neq \emptyset$ for every neighborhood $V$ of 0, and this happens if and only if $x \in A - V$ for every such $V$. Since $-V$ is a neighborhood of 0 if and only if $V$ is one, the proof is complete.

(b) Take $a \in \widetilde{A}, b \in \widetilde{B}$; let $W$ be a neighborhood of $a + b$. There are neighborhoods $W_1$ and $W_2$ of $a$ and $b$ such that $W_1 + W_2 \subseteq W$. There exist $x \in A \cap W_1$ and $y \in B \cap W_2$, since $a \in \widetilde{A}$ and $b \in \widetilde{B}$. Then $x + y$ lies in $(A + B) \cap W$, so that this intersection is not empty. Consequently, $a + b \in \widetilde{A + B}$.

(c) Suppose $\alpha$ and $\beta$ are scalars. By the proposition in Section 1.7, $a\bar{Y} = \bar{xY}$ if $\alpha \neq 0$; if $\alpha = 0$, these two sets are obviously equal. Hence it follows from (b) that

$$\alpha \bar{Y} + \beta \bar{Y} = \bar{xY} + \bar{yY} \subseteq \bar{\alpha Y} + \bar{\beta Y} \subseteq \bar{Y};$$

the assumption that $Y$ is a subspace was used in the last inclusion.

The proofs that convex sets have convex closures and that balanced sets have balanced closures are so similar to this proof of (c) that we shall omit them from (d) and (e).

(d) Since $C^\circ \subseteq C$ and $C$ is convex, we have

$$tC^\circ + (1 - t)C^\circ \subseteq C$$

if $0 < t < 1$. The two sets on the left are open; hence so is their sum. Since every open subset of $C$ is a subset of $C^\circ$, it follows that $C^\circ$ is convex.

(e) If $0 < |\alpha| < 1$, then $\alpha B^\circ = (\alpha B)^\circ$, since $x \mapsto \alpha x$ is a homeomorphism. Hence $\alpha B^\circ \subseteq \alpha B \subseteq B$, since $B$ is balanced. But $\alpha B^\circ$ is open. So $\alpha B^\circ \subseteq B^\circ$. If $B^\circ$ contains the origin, then $\alpha B^\circ \subseteq B^\circ$ even for $\alpha = 0$.

(f) Let $V$ be a neighborhood of 0. By Theorem 1.11, $\bar{W} \subseteq V$ for some neighborhood $W$ of 0. Since $E$ is bounded, $E \subseteq tW$ for all sufficiently large $t$. For these $t$, we have $\bar{E} \subseteq t\bar{W} \subseteq tV$.

1.14 Theorem In a topological vector space $X$,

(a) every neighborhood of 0 contains a balanced neighborhood of 0, and

(b) every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

PROOF. (a) Suppose $U$ is a neighborhood of 0 in $X$. Since scalar multiplication is continuous, there is a $\delta > 0$ and there is a neighborhood $V$ of 0 in $X$ such that $\alpha V \subseteq U$ whenever $|\alpha| < \delta$. Let $W$ be the union of all these sets $\alpha V$. Then $W$ is a neighborhood of 0, $W$ is balanced, and $W \subseteq U$. 

///
(b) Suppose $U$ is a convex neighborhood of 0 in $X$. Let $A = \bigcap \alpha U$, where $\alpha$ ranges over the scalars of absolute value 1. Choose $W$ as in part (a). Since $W$ is balanced, $\alpha^{-1}W = W$ when $|\alpha| = 1$; hence $W \subseteq \alpha U$. Thus $W \subseteq A$, which implies that the interior $A^\circ$ of $A$ is a neighborhood of 0. Clearly $A^\circ \subseteq U$. Being an intersection of convex sets, $A$ is convex; hence so is $A^\circ$. To prove that $A^\circ$ is a neighborhood with the desired properties, we have to show that $A^\circ$ is balanced; for this it suffices to prove that $A$ is balanced. Choose $r$ and $\beta$ so that $0 \leq r \leq 1$, $|\beta| = 1$. Then

$$r\beta A = \bigcap_{|\alpha| = 1} r\beta \alpha U = \bigcap_{|\alpha| = 1} r\alpha U.$$ 

Since $\alpha U$ is a convex set that contains 0, we have $r\alpha U \subseteq \alpha U$. Thus $r\beta A \subseteq A$, which completes the proof. 

Theorem 1.14 can be restated in terms of local bases. Let us say that a local base $\mathcal{B}$ is balanced if its members are balanced sets, and let us call $\mathcal{B}$ convex if its members are convex sets.

**Corollary**

(a) Every topological vector space has a balanced local base.

(b) Every locally convex space has a balanced convex local base.

Recall also that Theorem 1.11 holds for each of these local bases.

1.15 **Theorem** Suppose $V$ is a neighborhood of 0 in a topological vector space $X$.

(a) If $0 < r_1 < r_2 < \cdots$ and $r_n \to \infty$ as $n \to \infty$, then

$$X = \bigcup_{n=1}^{\infty} r_n V.$$ 

(b) Every compact subset $K$ of $X$ is bounded.

(c) If $\delta_1, \delta_2, \cdots$ and $\delta_n \to 0$ as $n \to \infty$, and if $V$ is bounded, then the collection

$$\{\delta_n V : n = 1, 2, 3, \ldots\}$$

is a local base for $X$.

**Proof.** (a) Fix $x \in X$. Since $\alpha \to \alpha x$ is a continuous mapping of the scalar field into $X$, the set of all $\alpha$ with $\alpha x \in V$ is open, contains 0, hence contains $1/r_n$ for all large $n$. Thus $(1/r_n)x \in V$, or $x \in r_n V$, for large $n$. 
(b) Let $W$ be a balanced neighborhood of 0 such that $W \subset V$. By (a),

$$K \subset \bigcup_{n=1}^{\infty} nW.$$  

Since $K$ is compact, there are integers $n_1 < \cdots < n_s$ such that

$$K \subset n_1 W \cup \cdots \cup n_s W = n_s W.$$  

The equality holds because $W$ is balanced. If $t > n_s$, it follows that $K \subset tW \subset tV$.

(c) Let $U$ be a neighborhood of 0 in $X$. If $V$ is bounded, there exists $s > 0$ such that $V \subset tU$ for all $t > s$. If $n$ is so large that $s \delta_n < 1$, it follows that $V \subset (1/\delta_n)U$. Hence $U$ actually contains all but finitely many of the sets $\delta_n V$.

Linear Mappings

1.16 Definitions  When $X$ and $Y$ are sets, the symbol

$$f: X \to Y$$

will mean that $f$ is a mapping of $X$ into $Y$. If $A \subset X$ and $B \subset Y$, the image $f(A)$ of $A$ and the inverse image or preimage $f^{-1}(B)$ of $B$ are defined by

$$f(A) = \{f(x) : x \in A\}, \quad f^{-1}(B) = \{x : f(x) \in B\}.$$  

Suppose now that $X$ and $Y$ are vector spaces over the same scalar field. A mapping $\Lambda: X \to Y$ is said to be linear if

$$\Lambda(\alpha x + \beta y) = \alpha \Lambda x + \beta \Lambda y$$

for all $x$ and $y$ in $X$ and all scalars $\alpha$ and $\beta$. Note that one often writes $\Lambda x$, rather than $\Lambda(x)$, when $\Lambda$ is linear.

Linear mappings of $X$ into its scalar field are called linear functionals. For example, the multiplication operators $M_a$ of Section 1.7 are linear, but the translation operators $T_a$ are not, except when $a = 0$.

Here are some properties of linear mappings $\Lambda: X \to Y$ whose proofs are so easy that we omit them; it is assumed that $A \subset X$ and $B \subset Y$:

(a) $\Lambda 0 = 0$.
(b) If $A$ is a subspace (or a convex set, or a balanced set) the same is true of $\Lambda(A)$.
(c) If $B$ is a subspace (or a convex set, or a balanced set) the same is true of $\Lambda^{-1}(B)$. 
In particular, the set

\[ \Lambda^{-1}(\{0\}) = \{ x \in X : \Lambda x = 0 \} = \mathcal{N}(\Lambda) \]

is a subspace of \( X \), called the null space of \( \Lambda \).

We now turn to continuity properties of linear mappings.

1.17 Theorem Let \( X \) and \( Y \) be topological vector spaces. If \( \Lambda : X \to Y \) is linear and continuous at 0, then \( \Lambda \) is continuous. In fact, \( \Lambda \) is uniformly continuous, in the following sense: To each neighborhood \( W \) of 0 in \( Y \) corresponds a neighborhood \( V \) of 0 in \( X \) such that

\[ y - x \in V \implies \Lambda y - \Lambda x \in W. \]

Proof. Once \( W \) is chosen, the continuity of \( \Lambda \) at 0 shows that \( \Lambda V \subset W \) for some neighborhood \( V \) of 0. If now \( y - x \in V \), the linearity of \( \Lambda \) shows that \( \Lambda y - \Lambda x = \Lambda(y - x) \in W \). Thus \( \Lambda \) maps the neighborhood \( x + V \) of \( x \) into the preassigned neighborhood \( \Lambda x + W \) of \( \Lambda x \), which says that \( \Lambda \) is continuous at \( x \).

1.18 Theorem Let \( \Lambda \) be a linear functional on a topological vector space \( X \). Assume \( \Lambda x \neq 0 \) for some \( x \in X \). Then each of the following four properties implies the other three:

(a) \( \Lambda \) is continuous.
(b) The null space \( \mathcal{N}(\Lambda) \) is closed.
(c) \( \mathcal{N}(\Lambda) \) is not dense in \( X \).
(d) \( \Lambda \) is bounded in some neighborhood \( V \) of 0.

Proof. Since \( \mathcal{N}(\Lambda) = \Lambda^{-1}(\{0\}) \) and \( \{0\} \) is a closed subset of the scalar field \( \Phi \), (a) implies (b). By hypothesis, \( \mathcal{N}(\Lambda) \neq X \). Hence (b) implies (c).

Assume (c) holds; i.e., assume that the complement of \( \mathcal{N}(\Lambda) \) has nonempty interior. By Theorem 1.14,

\[ (x + V) \cap \mathcal{N}(\Lambda) = \emptyset \]

for some \( x \in X \) and some balanced neighborhood \( V \) of 0. Then \( \Lambda V \) is a balanced subset of the field \( \Phi \). Thus either \( \Lambda V \) is bounded, in which case (d) holds, or \( \Lambda V = \Phi \). In the latter case, there exists \( y \in V \) such that \( \Lambda y = -\Lambda x \), and so \( x + y \in \mathcal{N}(\Lambda) \), in contradiction to (1). Thus (c) implies (d).

Finally, if (d) holds then \( |\Lambda x| < M \) for all \( x \) in \( V \) and for some \( M < \infty \). If \( r > 0 \) and if \( W = (r/M)V \), then \( |\Lambda x| < r \) for every \( x \) in \( W \). Hence \( \Lambda \) is continuous at the origin. By Theorem 1.17, this implies (a).