This is the first modern treatment of orthogonal polynomials from the viewpoint of special functions. The coverage is encyclopedic, including classical topics such as Jacobi, Hermite, Laguerre, Hahn, Charlier and Meixner polynomials as well as those, e.g. Askey–Wilson and Al-Salam–Chihara polynomial systems, discovered over the last 50 years: multiple orthogonal polynomials are discussed for the first time in book form. Many modern applications of the subject are dealt with, including birth and death processes, integrable systems, combinatorics, and physical models. A chapter on open research problems and conjectures is designed to stimulate further research on the subject.

Exercises of varying degrees of difficulty are included to help the graduate student and the newcomer. A comprehensive bibliography rounds off the work, which will be valued as an authoritative reference and for graduate teaching, in which role it has already been successfully class-tested.
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Classical and Quantum Orthogonal Polynomials in One Variable

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With two chapters by
Walter Van Assche
Catholic University of Leuven
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**Foreword**

**Preface**

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Foreword

There are a number of ways of studying orthogonal polynomials. Gabor Szegő’s book “Orthogonal Polynomials” (Szegő, 1975) had two main topics. Most of this book dealt with polynomials which are orthogonal on the real line, with a chapter on polynomials orthogonal on the unit circle and a short chapter on polynomials orthogonal on more general curves. About two-thirds of Szegő’s book deals with the classical orthogonal polynomials of Jacobi, Laguerre and Hermite, which are orthogonal with respect to the beta, gamma and normal distributions, respectively. The rest deals with more general sets of orthogonal polynomials, some general theory, and some asymptotics.

Barry Simon has recently written a very long book on polynomials orthogonal on the unit circle, (Simon, 2004). His book has very little on explicit examples, so its connection with Szegő’s book is mainly in the general theory, which has been developed much more deeply than it had been in 1938 when Szegő’s book appeared.

The present book, by Mourad Ismail, complements Szegő’s book in a different way. It primarily deals with specific sets of orthogonal polynomials. These include the classical polynomials mentioned above and many others. The classical polynomials of Jacobi, Laguerre and Hermite satisfy second-order linear homogeneous differential equations of the form

\[ a(x)y''(x) + b(x)y'(x) + \lambda_n y(x) = 0 \]

where \( a(x) \) and \( b(x) \) are polynomials of degrees 2 and 1, respectively, which are independent of \( n \), and \( \lambda_n \) is independent of \( x \). They have many other properties in common. One is that the derivative of \( p_n(x) \) is a constant times \( q_{n-1}(x) \) where \( \{p_n(x)\} \) is in one of these classes of polynomials and \( \{q_n(x)\} \) is also. These are the only sets of orthogonal polynomials with the property that their derivatives are also orthogonal.

Many of the classes of polynomials studied in this book have a similar nature, but with the derivative replaced by another operator. The first operator which was used is

\[ \Delta f(x) = f(x + 1) - f(x), \]

a standard form of a difference operator. Later, a \( q \)-difference operator was used

\[ D_q f(x) = [f(qx) - f(x)]/[qx - x]. \]
Still later, two divided difference operators were introduced. The orthogonal polynomials which arise when the \( q \)-divided difference operator is used contain a set of polynomials introduced by L. J. Rogers in a remarkable series of papers which appeared in the 1890s. One of these sets of polynomials was used to derive what we now call the Rogers–Ramanujan identities. However, the orthogonality of Rogers’s polynomials had to wait decades before it was found. Other early work which leads to polynomials in the class of these generalized classical orthogonal polynomials was done by Chebyshev, Markov and Stieltjes.

To give an idea about the similarities and differences of the classical polynomials and some of the extensions, consider a set of polynomials called ultraspherical or Gegenbauer polynomials, and the extension Rogers found. Any set of polynomials which is orthogonal with respect to a positive measure on the real line satisfies a three term recurrence relation which can be written in a number of equivalent ways. The ultraspherical polynomials \( C_\nu^n(x) \) are orthogonal on \((-1, 1)\) with respect to \((1 - x^2)^{\nu-1/2}\). Their three-term recurrence relation is

\[
2(n + \nu) x C_\nu^n(x) = (n + 1) C_\nu^{n+1}(x) + (n + 2\nu - 1) C_\nu^{n-1}(x)
\]

The three-term recurrence relation for the continuous \( q \)-ultraspherical polynomials of Rogers satisfy a similar recurrence relation with every \((n + a)\) replaced by \(1 - q^{n+a}\). That is a natural substitution to make, and when the recurrence relation is divided by \(1 - q\), letting \( q \) approach 1 gives the ultraspherical polynomials in the limit.

Both of these sets of polynomials have nice generating functions. For the ultraspherical polynomials one nice generating function is

\[
(1 - 2x r + r^2)^{-\nu} = \sum_{n=0}^{\infty} C_\nu^n(x) r^n
\]

The extension of this does not seem quite as nice, but when the substitution \( x = \cos \theta \) is used on both, they become similar enough for one to guess what the left-hand side should be. Before the substitution it is

\[
\prod_{n=0}^{\infty} \frac{(1 - 2x q^{\nu+n} r + q^{2\nu+2n} r^2)}{(1 - 2x q^{\nu} r + q^{2\nu} r^2)} = \sum_{n=0}^{\infty} C_n(x; q^\nu | q) r^n.
\]

The weight function is a completely different story. To see this, it is sufficient to state it:

\[
w(x, q^\nu) = (1 - x^2)^{-1/2} \prod_{n=0}^{\infty} \frac{(1 - (2x^2 - 1) q^n + q^{2n})}{(1 - (2x^2 - 1) q^{\nu+n} + q^{2\nu+2n})}.
\]

These polynomials of Rogers were rediscovered about 1940 by two mathematicians, (Feldheim, 1941b) and (Lanzewizky, 1941). Enough had been learned about orthogonal polynomials by then for them to know they had sets of orthogonal polynomials, but neither could find the orthogonality relation. One of these two mathematicians, E. Feldheim, lamented that he was unable to find the orthogonality relation. Stieltjes and Markov had found theorems which would have allowed Feldheim to work out the orthogonality relation, but there was a war going on when Feldheim did his work and he was unaware of this old work of Stieltjes and Markov. The limiting case when
$\nu \to \infty$ gives what are called the continuous $q$-Hermite polynomials. It was these polynomials which Rogers used to derive the Rogers-Ramanujan identities.

Surprisingly, these polynomials have recently come up in a very attractive problem in probability theory which has no $q$ in the statement of the problem. See Bryc (Bryc, 2001) for this work.

Stieltjes solved a minimum problem which can be considered as coming from one dimensional electrostatics, and in the process found the discriminant for Jacobi polynomials. The second-order differential equation they satisfy played an essential role. When I started to study special functions and orthogonal polynomials, it seemed that the only orthogonal polynomials which satisfied differential equations nice enough to be useful were Jacobi, Laguerre and Hermite. For a few classes of orthogonal polynomials nice enough differential equations existed, but they were not well known. Now, thanks mainly to a conjecture of G. Freud which he proved in two very special cases, and work by quite a few people including Nevai and some of his students, we know that nice enough differential equations exist for polynomials orthogonal with respect to $\exp(-v(x))$ when $v(x)$ is smooth enough. The work of Stieltjes can be partly extended to this much wider class of orthogonal polynomials. Some of this is done in Chapter 3.

Chapter 4 deals with the classical polynomials. For Hermite polynomials there is an explicit expression for the analogue of the Poisson kernel for Fourier series which was found by Mehler in the 19th century. An important multivariable extension of this formula found independently by Kibble and Slepian is in Chapter 4. Chapter 5 contains some information about the Pollaczek polynomials on the unit interval. Their recurrence relation is a slight variant of the one for ultraspherical polynomials listed above. The weight function is drastically different, having infinitely many point masses outside the interval where the absolutely continuous part is supported or vanishing very rapidly at one or both of the end points of the interval supporting the absolutely continuous part of the orthogonality measure.

Chapter 6 deals with extensions of the classical orthogonal polynomials whose weight function is discrete. Here the classical discriminant seemingly cannot be found in a useful form, but a variant of it has been computed for the Hahn polynomials. This extends the result of Stieltjes on the discriminant for Jacobi polynomials. Hahn polynomials extend Jacobi polynomials and are orthogonal with respect to the hypergeometric distribution. Transformations of them occur in the quantum theory of angular momentum and they and their duals occur in some settings of coding theory.

The polynomials considered in the first 10 chapters which have explicit formulas are given as generalized hypergeometric series. These are series whose term ratio is a rational function of $n$. In Chapters 11 to 19 a different setting occurs, that of basic hypergeometric series. These are series whose term ratio is a rational function of $q^n$.

In the 19th century Markov and Stieltjes found examples of orthogonal polynomials which can be written as basic hypergeometric series and found an explicit orthogonality relation. As mentioned earlier, Rogers also found some polynomials which are orthogonal and can be given as basic hypergeometric series, but he was unaware they were orthogonal. A few other examples were found before Wolfgang Hahn
wrote a major paper, (Hahn, 1949b) in which he found basic hypergeometric extensions of the classical polynomials and the discrete ones up to the Hahn polynomial level. There is one level higher than this where orthogonal polynomials exist which have properties very similar to many of those known for the classical orthogonal polynomials. In particular, they satisfy a second-order divided $q$-difference equation and this divided $q$-difference operator applied to them gives another set of orthogonal polynomials. When this was first published, the polynomials were treated directly without much motivation. Here simpler cases are done first and then a boot-strap argument allows one to obtain more general polynomials, eventually working up to the most general classical type sets of orthogonal polynomials.

The most general of these polynomials has four free parameters in addition to the $q$ of basic hypergeometric series. When three of the parameters are held fixed and the fourth is allowed to vary, the coefficients which occur when one is expanded in terms of the other are given as products. The resulting identity contains a very important transformation formula between a balanced $4\phi_3$ and a very-well-poised $8\phi_7$ which Watson found in the 1920s as the master identity which contains the Rogers-Ramanujan identities as special cases and many other important formulas.

There are many ways to look at this identity of Watson, and some of these ways lead to interesting extensions. When three of the four parameters are shifted and this connection problem is solved, the coefficients are single sums rather than the double sums which one expects. At present we do not know what this implies, but surprising results are usually important, even if it takes a few decades to learn what they imply.

The fact that there are no more classical type polynomials beyond those mentioned in the last paragraph follows from a theorem of Leonard (Leonard, 1982). This theorem has been put into a very attractive setting by Terwilliger, some of whose work has been summarized in Chapter 20. However, that is not the end since there are biorthogonal rational functions which have recently been discovered. Some of this work is contained in Chapter 18. There is even one higher level than basic hypergeometric functions, elliptic hypergeometric functions. Gasper and Rahman have included a chapter on them in (Gasper & Rahman, 2004).

Chapters 22 and 23 were written by Walter Van Assche. The first is on the Riemann-Hilbert method of studying orthogonal polynomials. This is a very powerful method for deriving asymptotics of wide classes of orthogonal polynomials. The second chapter is on multiple orthogonal polynomials. These are polynomials in one variable which are orthogonal with respect to $r$ different measures. The basic ideas go back to the 19th century, but except for isolated work which seems to start with Angelesco in 1919, it has only been in the last 20 or so years that significant work has been done on them.

There are other important results in this book. One which surprised me very much is the $q$-version of Airy functions, at least as the two appear in asymptotics. See, for example, Theorem 21.7.3.
When I started to work on orthogonal polynomials and special functions, I was told by a number of people that the subject was out-of-date, and some even said dead. They were wrong. It is alive and well. The one variable theory is far from finished, and the multivariable theory has grown past its infancy but not enough for us to be able to predict what it will look like in 2100.

Madison, WI
April 2005

Richard A. Askey
Preface

I first came across the subject of orthogonal polynomials when I was a student at Cairo University in 1964. It was part of a senior-level course on special functions taught by the late Professor Foad M. Ragab. The instructor used his own notes, which were very similar in spirit to the way Rainville treated the subject. I enjoyed Ragab’s lectures and, when I started graduate school in 1968 at the University of Alberta, I was fortunate to work with Waleed Al-Salam on special functions and $q$-series. Jerry Fields taught me asymptotics and was very generous with his time and ideas. In the late 1960s, courses in special functions were a rarity at North American universities and have been replaced by Bourbaki-type mathematics courses. In the early 1970s, Richard Askey emerged as the leader in the area of special functions and orthogonal polynomials, and the reader of this book will see the enormous impact he made on the subject of orthogonal polynomials. At the same time, George Andrews was promoting $q$-series and their applications to number theory and combinatorics. So when Andrews and Askey joined forces in the mid-1970s, their combined expertise advanced the subject in leaps and bounds. I was very fortunate to have been part of this group and to participate in these developments. My generation of special functions / orthogonal polynomials people owes Andrews and Askey a great deal for their ideas which fueled the subject for a while, for the leadership role they played, and for taking great care of young people.

This book project started in the early 1990s as lecture notes on $q$-orthogonal polynomials with the goal of presenting the theory of the Askey–Wilson polynomials in a way suitable for use in the classroom. I taught several courses on orthogonal polynomials at the University of South Florida from these notes, which evolved with time. I later realized that it would be better to write a comprehensive book covering all known systems of orthogonal polynomials in one variable. I have attempted to include as many applications as possible. For example, I included treatments of the Toda lattice and birth and death processes. Applications of connection relations for $q$-polynomials to the evaluation of integrals and the Rogers–Ramanujan identities are also included. To the best of my knowledge, my treatment of associated orthogonal polynomials is a first in book form. I tried to include all systems of orthogonal polynomials but, in order to get the book out in a timely fashion, I had to make some compromises. I realized that the chapters on Riemann–Hilbert problems and multiple orthogonal polynomials should be written by an expert on the subject, and
Preface

Walter Van Assche kindly agreed to write this material. He wrote Chapters 22 and 23, except for §22.8. Due to the previously mentioned time constraints, I was unable to treat some important topics. For example, I covered neither the theories of matrix orthogonal polynomials developed by Antonio Durán, Yuan Xu and their collaborators, nor the recent interesting explicit systems of Grünbaum and Tirao and of Durán and Grünbaum. I hope to do so if the book has a second edition. Regrettably, neither the Sobolov orthogonal polynomials nor the elliptic biorthogonal rational functions are treated.

Szegő’s book on orthogonal polynomials inspired generations of mathematicians. The character of this volume is very different from Szegő’s book. We are mostly concerned with the special functions aspects of orthogonal polynomials, together with some general properties of orthogonal polynomial systems. We tried to minimize the possible overlap with Szegő’s book. For example, we did not treat the refined bounds on zeros of Jacobi, Hermite and Laguerre polynomials derived in (Szegő, 1975) using Sturmian arguments. Although I tried to cover a broad area of the subject matter, the choice of the material is influenced by the author’s taste and personal bias.

Dennis Stanton has used parts of this book in a graduate-level course at the University of Minnesota and kindly supplied some of the exercises. His careful reading of the book manuscript and numerous corrections and suggestions are greatly appreciated. Thanks also to Richard Askey and Erik Koelink for reading the manuscript and providing a lengthy list of corrections and additional information. I am grateful to Paul Terwilliger for his extensive comments on §20.3.

I hope this book will be useful to students and researchers alike. It has a collection of open research problems in Chapter 24 whose goal is to challenge the reader’s curiosity. These problems have varying degrees of difficulty, and I hope they will stimulate further research in this area.

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Orlando, FL

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April 2005
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Preliminaries

In this chapter we collect results from linear algebra and real and complex analysis which we shall use in this book. We will also introduce the definitions and terminology used. Some special functions are also introduced in the present chapter, but the $q$-series and related material are not defined until Chapter 11. See Chapters 11 and 12 for $q$-series.

1.1 Hermitian Matrices and Quadratic Forms

Recall that a matrix $A = (a_{j,k})$, $1 \leq j, k \leq n$ is called Hermitian if

$$a_{j,k} = a_{k,j}, \quad 1 \leq j, k \leq n.$$  

(1.1.1)

We shall use the following inner product on the $n$-dimensional complex space $\mathbb{C}^n$,

$$(x, y) = \sum_{j=1}^{n} x_j \bar{y}_j, \quad x = (x_1, \ldots, x_n)^T, \quad y = (y_1, \ldots, y_n)^T,$$  

(1.1.2)

where $A^T$ is the transpose of $A$. Clearly

$$(x, y) = (\bar{y}, \bar{x}), \quad (a x, y) = a(x, y), \quad a \in \mathbb{C}. $$

Two vectors $x$ and $y$ are called orthogonal if $(x, y) = 0$. The adjoint $A^*$ of $A$ is the matrix satisfying

$$(A x, y) = (x, A^* y).$$  

(1.1.3)

It is easy to see that if $A = (a_{j,k})$ then $A^* = (\bar{a}_{k,j})$. Thus, $A$ is Hermitian if and only if $A^* = A$. The eigenvalues of Hermitian matrices are real. This is so since $A x = \lambda x$, $x \neq 0$ then

$$\lambda(x, x) = (A x, x) = (x, A^* x) = (x, \lambda x) = \bar{\lambda}(x, x).$$

Furthermore, the eigenvectors corresponding to distinct eigenvalues are orthogonal. This is the case because if $A x = \lambda_1 x$ and $A y = \lambda_2 y$ then

$$\lambda_1(x, y) = (A x, y) = (x, A y) = \lambda_2(x, y),$$

hence $(x, y) = 0$. 

1
Any Hermitian matrix generates a quadratic form

$$\sum_{j,k=1}^{n} a_{j,k} x_j x_k, \quad (1.1.4)$$

and conversely any quadratic form with $\overline{a_{j,k}} = a_{k,j}$ determines a Hermitian matrix $A$ through

$$\sum_{j,k=1}^{n} a_{j,k} x_j x_k = x^* A x = (Ax, x). \quad (1.1.5)$$

In an infinite dimensional Hilbert space $\mathcal{H}$, the adjoint is defined by (1.1.3) provided it holds for all $x, y \in \mathcal{H}$. A linear operator $A$ defined in $\mathcal{H}$ is called self-adjoint if $A = A^*$. On the other hand, $A$ is called symmetric if

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

whenever both sides are defined.

**Theorem 1.1.1** Assume that the entries of a matrix $A$ satisfy $|a_{j,k}| \leq M$ for all $j, k$ and that each row of $A$ has at most $\ell$ nonzero entries. Then all the eigenvalues of $A$ satisfy

$$|\lambda| \leq \ell M.$$ 

**Proof** Take $x$ to be an eigenvector of $A$ with an eigenvalue $\lambda$, and assume that $\|x\| = 1$. Observe that the Cauchy–Schwartz inequality implies

$$|\lambda|^2 = |\langle Ax, x \rangle|^2 = \left| \sum_{j,k=1}^{n} a_{j,k} x_j x_k \right|^2 \leq \|x\|^2 \sum_{j=1}^{n} \left| \sum_{k=1}^{n} a_{j,k} x_k \right|^2 \leq \ell^2 M^2.$$ 

Hence the theorem is proved. $\square$

A quadratic form (1.1.4) is **positive definite** if $\langle Ax, x \rangle > 0$ for any nonzero $x$. Recall that a matrix $U$ is unitary if $U^* = U^{-1}$. The spectral theorem for Hermitian matrices is:

**Theorem 1.1.2** For every Hermitian matrix $A$ there is a unitary matrix $U$ whose columns are the eigenvectors of $A$ such that

$$A = U^* \Lambda U, \quad (1.1.6)$$

and $\Lambda$ is the diagonal matrix formed by the corresponding eigenvalues of $A$.

For a proof see (Horn & Johnson, 1992). An immediate consequence of Theorem 1.1.2 is the following corollary.
1.2 Some Real and Complex Analysis

Corollary 1.1.3 The quadratic form (1.1.4) is reducible to a sum of squares,
\[ \sum_{j,k=1}^{n} a_{j,k} x_j x_k = \sum_{k=1}^{n} \lambda_k |y_k|^2, \]  
(1.1.7)
where \( y = Ux \), and \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \).

The following important characterization of positive definite forms follows from Corollary 1.1.3.

Theorem 1.1.4 The quadratic form (1.1.4)–(1.1.5) is positive definite if and only if the eigenvalues of \( A \) are positive.

We next state the Sylvester criterion for positive definiteness (Shilov, 1977), (Horn & Johnson, 1992).

Theorem 1.1.5 The quadratic form (1.1.5) is positive definite if and only if the principal minors of \( A \), namely
\[ \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}, \]  
(1.1.8)
are positive.

Recall that a matrix \( A = (a_{j,k}) \) is called \textbf{strictly diagonally dominant} if
\[ 2 |a_{j,j}| > \sum_{k=1}^{n} |a_{j,k}|. \]  
(1.1.9)

The following criterion for positive definiteness is in (Horn & Johnson, 1992, Theorem 6.1.10).

Theorem 1.1.6 Let \( A \) be \( n \times n \) matrix which is Hermitian, strictly diagonally dominant, and its diagonal entries are positive. Then \( A \) is positive definite.

1.2 Some Real and Complex Analysis

We need some standard results from real and complex analysis which we shall state without proofs and provide references to where proofs can be found. We shall normalize functions of bounded variations to be continuous on the right.

Theorem 1.2.1 (Helly’s selection principle) Let \( \{\psi_n(x)\} \) be a sequence of uniformly bounded nondecreasing functions. Then there is a subsequence \( \{\psi_{k_n}(x)\} \) which converges to a nondecreasing bounded function, \( \psi \). Moreover if for every \( n \) the moments \( \int_{\mathbb{R}} x^m d\psi_n(x) \) exist for all \( m, n = 0, 1, \ldots \), then the moments of \( \psi \) exist and \( \int_{\mathbb{R}} x^m d\psi_n(x) \) converges to \( \int_{\mathbb{R}} x^m d\psi(x) \). Furthermore if \( \{\psi_n(x)\} \) does not converge, then there are at least two such convergent subsequences.
For a proof we refer the reader to Section 3 of the introduction to Shohat and Tamarkin (Shohat & Tamarkin, 1950).

**Theorem 1.2.2 (Vitali)** Let \( \{f_n(z)\} \) be a sequence of functions analytic in a domain \( D \) and assume that \( f_n(z) \to f(z) \) pointwise in \( D \). Then \( f_n(z) \to f(z) \) uniformly in any subdomain bounded by a contour \( C \), provided that \( C \) is contained in \( D \).

A proof is in Titchmarsh (Titchmarsh, 1964, page 168).

We now briefly discuss the Lagrange inversion and state two useful identities that will be used in later chapters.

**Theorem 1.2.3 (Lagrange)** Let \( f(z) \) and \( \phi(z) \) be functions of \( z \) analytic on and inside a contour \( C \) containing the point \( a \) in its interior. Let \( t \) be such that \( |t\phi(z)| < |z-a| \) on the contour \( C \). Then the equation

\[
\zeta = a + t\phi(\zeta),
\]

regarded as an equation in \( \zeta \), has one root interior to \( C \); and further any function of \( \zeta \) analytic on the closure of the interior of \( C \) can be expanded as a power series in \( t \) by the formula

\[
f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left[ \frac{d^{n-1} f(x)[\phi(x)]^n}{dx^{n-1}} \right]_{x=a}.
\]

See Whittaker and Watson (Whittaker & Watson, 1927, §7.32), or Polya and Szegő (Pólya & Szegő, 1972, p. 145). An equivalent form is

\[
\frac{f(\zeta)}{1-t\phi'(\zeta)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ \frac{d^n f(x)[\phi(x)]^n}{dx^n} \right]_{x=a}
\]

Two important special cases are \( \phi(z) = e^z \), or \( \phi(z) = (1+z)^\beta \). These cases lead to:

\[
ea^z = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+n)^{n-1}}{n!} w^n, \quad w = z e^{-z},
\]

\[
(1+z)^\alpha = 1 + \alpha \sum_{n=1}^{\infty} \left( \frac{\alpha+\beta n-1}{n-1} \right) \frac{w^n}{n!}, \quad w = z(1+z)^{-\beta}.
\]

We say that (Olver, 1974)

\[
f(z) = O(g(z)), \quad \text{as } z \to a,
\]

if \( f(z)/g(z) \) is bounded in a neighborhood of \( z = a \). On the other hand we write

\[
f(z) = o(g(z)), \quad \text{as } z \to a
\]

if \( f(z)/g(z) \to 0 \) as \( z \to a \).

A very useful method to determine the large \( n \) behavior of orthogonal polynomials \( \{p_n(x)\} \) is Darboux’s asymptotic method.
Theorem 1.2.4 Let \( f(z) \) and \( g(z) \) be analytic in \( \{ z : |z| < r \} \) and assume that

\[
f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad g(z) = \sum_{n=0}^{\infty} g_n z^n, \quad |z| < r.
\] (1.2.6)

If \( f - g \) is continuous on the closed disc \( \{ z : |z| \leq r \} \) then

\[
f_n = g_n + o(r^{-n}).
\] (1.2.7)

This form of Darboux’s method is in (Olver, 1974, Ch. 8) and, in view of Cauchy’s formulas, is just a restatement of the Riemann–Lebesgue lemma. For a given function \( f, g \) is called a comparison function. Another proof of Darboux’s lemma is in (Knuth & Wilf, 1989).

In order to apply Darboux’s method to a sequence \( \{ f_n \} \) we need first to find a generating function for the \( f_n \)’s, that is, find a function whose Taylor series expansion around \( z = 0 \) has coefficients \( c_n f_n \), for some simple sequence \( \{ c_n \} \). In this work we pay particular attention to generating functions of orthogonal polynomials and Darboux’s method will be used to derive asymptotic expansions for some of the orthogonal polynomials treated in this work. The recent work (Wong & Zhao, 2005) shows how Darboux’s method can be used to derive uniform asymptotic expansions. This is a major simplification of the version in (Fields, 1967). Wang and Wong developed a discrete version of the Liouville–Green approximation (WKB) in (Wang & Wong, 2005a). This gives uniform asymptotic expansions of a basis of solutions of three-term recurrence relations. This technique is relevant, because all orthogonal polynomials satisfy three-term recurrence relations.

The Perron–Stieltjes inversion formula, see (Stone, 1932, Lemma 5.2), is

\[
F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}, \quad z \notin \mathbb{R}
\] (1.2.8)

if and only if

\[
\mu(t) - \mu(s) = \lim_{\epsilon \to 0^+} \int_{s}^{t} \frac{F(x - i\epsilon) - F(x + i\epsilon)}{2\pi i} \, dx.
\] (1.2.9)

The above inversion formula enables us to recover \( \mu \) from knowing its Stieltjes transform \( F(z) \).

Remark 1.2.1 It is clear that if \( \mu \) has an isolated atom \( u \) at \( x = a \) then \( z = a \) will be a pole of \( F \) with residue equal to \( u \). Conversely, the poles of \( F \) determine the location of the isolated atoms of \( \mu \) and the residues determine the corresponding masses. Formula (1.2.9) captures this behavior and reproduces the residue at an isolated singularity.

Remark 1.2.2 Formula (1.2.9) shows that the absolutely continuous component of \( \mu \) is given by

\[
\mu'(x) = \left[ F(x - i0^+) - F(x + i0^+) \right] / (2\pi i).
\] (1.2.10)
Preliminaries

An analytic function defined on a closed disc is bounded and its absolute value attains its maximum on the boundary.

**Definition 1.2.1** Let \(f\) be an entire function. The maximum modulus is

\[
M(r; f) := \sup \{|f(z)| : |z| \leq r\}, \quad r > 0.
\]  

(1.2.11)

The order of \(f, \rho(f)\) is defined by

\[
\rho(f) := \limsup_{r \to \infty} \frac{\ln M(r, f)}{\ln r}.
\]

(1.2.12)

**Theorem 1.2.5** (Boas, Jr., 1954) If \(\rho(f)\) is finite and is not equal to a positive integer, then \(f\) has infinitely many zeros.

If \(f\) has finite order, its type \(\sigma\) is

\[
\sigma = \inf \{K : M(r) < \exp(Kr^\rho)\}.
\]

(1.2.13)

For an entire function of finite order and type we define the Phragmén–Lindelöf indicator \(h(\theta)\) as

\[
h(\theta) = \lim_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r^\rho}.
\]

(1.2.14)

Consider the infinite product

\[
P = \prod_{n=1}^{\infty} (1 + a_n).
\]

(1.2.15)

We say the \(P\) converges to \(\ell, \ell \neq 0\), if

\[
\lim_{m \to \infty} \prod_{n=1}^{m} (1 + a_n) = \ell.
\]

If \(\ell = 0\) we say \(P\) diverges to zero. One can prove, see (Rainville, 1960, Chapter 1), that \(a_n \to 0\) is necessary for \(P\) to converge. Similarly, one can define absolute convergence of infinite products. When \(a_n = a_n(z)\) are functions of \(z\), say, we say that \(P\) converges uniformly in a domain \(D\) if the partial products

\[
\prod_{n=1}^{m} (1 + a_n(z))
\]

converge uniformly in \(D\) to a function with no zeros in \(D\).

**Definition 1.2.2** Given a set of distinct points \(\{x_j : 1 \leq j \leq n\}\), the Lagrange fundamental polynomial \(\ell_k(x)\) is

\[
\ell_k(x) = \prod_{\substack{j=1 \atop j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)} = \frac{S_n(x)}{S_n'(x_k)(x - x_k)}, \quad 1 \leq k \leq n,
\]

(1.2.16)
1.2 Some Real and Complex Analysis

where \( S_n(x) = \prod_{1}^{n} (x - x_j) \). The Lagrange interpolation polynomial of a function \( f(x) \) at the nodes \( x_1, \ldots, x_n \) is the unique polynomial \( L(x) \) of degree \( n - 1 \) such that \( f(x_j) = L(x_j) \).

It is easy to see that \( L(x) \) in Definition 1.2.2 is

\[
L(x) = \sum_{k=1}^{n} \ell_k(x) f(x_k) = \sum_{k=1}^{n} f(x_k) \frac{S_n(x)}{S_n'(x_k)(x-x_k)}. \tag{1.2.17}
\]

**Theorem 1.2.6 (Poisson Summation Formula)** Let \( f \in L_1(\mathbb{R}) \) and \( F \) be its Fourier transform,

\[ F(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x)e^{-ixt} \, dx, \quad t \in \mathbb{R}. \]

Then

\[
\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(x)e^{-inx} \, dx.
\]

For a proof, see (Zygmund, 1968, §II.13).

**Theorem 1.2.7** Given two differential equations in the form

\[
\frac{d^2u}{dz^2} + f(z)u(z) = 0, \quad \frac{d^2v}{dz^2} + g(z)v(z) = 0,
\]

then \( y = uv \) satisfies

\[
\frac{d}{dz} \left\{ \frac{y'' + 2(f + g)y' + (f' + g')y}{f - g} \right\} + (f - g)y = 0, \quad \text{if } f \neq g \tag{1.2.18}
\]

\[
y'' + 4fy' + 2f'y = 0, \quad \text{if } f = g. \tag{1.2.19}
\]

A proof of Theorem 1.2.6 is in Watson (Watson, 1944, §5.4), where he attributes the theorem to P. Appell.

**Lemma 1.2.8** Let \( y = y(x) \) satisfy the differential equation

\[
\phi(x)y''(x) + y(x) = 0, \quad a < x < b \tag{1.2.20}
\]

where \( \phi(x) > 0 \), and \( \phi'(x) \) is positive (negative) and continuous on \((a, b)\). Then the successive relative maxima of \(|y|\) increase (decrease) with \( x \) in \((a, b)\) if \( \phi \) increases (decreases) on \((a, b)\).

**Proof** Let

\[
f(x) := \{y(x)\}^2 + \phi(x)\{y'(x)\}^2, \tag{1.2.21}
\]

so that \( f(x) = \{y(x)\}^2 \) if \( y'(x) = 0 \). Clearly

\[
f'(x) = y'(x)\{2y(x) + \phi'(x)y'(x) + 2\phi(x)y''(x)\}
\]

\[
= \phi'(x)\{y'(x)\}^2.
\]
Thus \( \text{sign } f'(x) = \text{sign } \phi' \) in between the consecutive successive maxima of \(|y|\) and the result follows.

1.3 Some Special Functions

Standard references in this area are (Andrews et al., 1999), (Bailey, 1935), (Rainville, 1960), (Erdélyi et al., 1953b), (Slater, 1964), (Whittaker & Watson, 1927).

The gamma and beta functions are probably the most important functions in mathematics beyond the exponential and logarithmic functions. Recall that

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \text{Re } z > 0,
\]

(1.3.1)

\[
B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad \text{Re } x > 0, \quad \text{Re } y > 0.
\]

(1.3.2)

They are related through

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

(1.3.3)

The functional equation

\[
\Gamma(z+1) = z\Gamma(z)
\]

(1.3.4)

extends the gamma function to a meromorphic function with poles at \( z = 0, -1, \ldots, \) and also extends \( B(x, y) \) to a meromorphic function of \( x \) and \( y \). The Mittag–Leffler expansion for \( \Gamma'/\Gamma \) is (Whittaker & Watson, 1927, §12.3)

\[
\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left[ \frac{1}{z+n} - \frac{1}{n} \right],
\]

(1.3.5)

where \( \gamma \) is the Euler constant, (Rainville, 1960, §7).

The shifted factorial is

\[
(a)_0 := 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n > 0,
\]

(1.3.6)

hence (1.3.4) gives

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.
\]

(1.3.7)

The shifted factorial is also called Pochhammer symbol. Note that (1.3.7) is meaningful for any complex \( n \), when \( a+n \) is not a pole of the gamma function. The gamma function and the shifted factorial satisfy the duplication formulas

\[
\Gamma(2z) = 2^{2z-1} \Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}, \quad (2a)_2n = 2^{2n}(a)_{n}(a + 1/2)_n.
\]

(1.3.8)

We also have the reflection formula

\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.
\]

(1.3.9)

We define the multishifted factorial as

\[
(a_1, \cdots, a_m)_n = \prod_{j=1}^{m} (a_j)_n.
\]
Some useful identities are
\[(a)_m (a + m)_n = (a)_{m+n}, \quad (a)_{N-k} = \frac{(a)_N (-1)^k}{(-a + N + 1)_k}. \tag{1.3.10}\]

A hypergeometric series is
\[\sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n} \frac{z^n}{n!}. \tag{1.3.11}\]

If one of the numerator parameters is a negative integer, say \(-k\), then the series \((1.3.11)\) becomes a finite sum, \(0 \leq n \leq k\) and the \(rF_s\) series is called terminating.

As a function of \(z\) nonterminating series is entire if \(r \leq s\), is analytic in the unit disc if \(r = s+1\). The hypergeometric function \(2F_1(a, b; c; z)\) satisfies the hypergeometric differential equation
\[z(1-z) \frac{d^2y}{dz^2} + [c - (a + b + 1)] \frac{dy}{dz} - aby = 0. \tag{1.3.12}\]

The confluent hypergeometric function (Erdélyi et al., 1953b, §6.1)
\[\Phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \tag{1.3.13}\]

satisfies the differential equation
\[z \frac{d^2y}{dz^2} + (c - z) \frac{dy}{dz} - ay = 0, \tag{1.3.14}\]
and \(\lim_{b \to \infty} 2F_1(a, b; cz/b) = F_1(a; c; z)\). The Tricomi \(\Psi\) function is a second linear independent solution of \((1.3.14)\) and is defined by (Erdélyi et al., 1953b, §6.5)
\[\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} \Phi(a-c+1, 2-c; x). \tag{1.3.15}\]

The function of \(\Psi\) has the integral presentation (Erdélyi et al., 1953a, §6.5)
\[\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1 + t)^{c-a-1} dt, \tag{1.3.16}\]
for \(\Re a > 0, \Re x > 0\).

The Bessel function \(J_\nu\) and the modified Bessel function \(I_\nu\), (Watson, 1944) are
\[J_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{\nu+2n}}{\Gamma(n + \nu + 1) n!}, \tag{1.3.17}\]
\[I_{\nu}(z) = \sum_{n=0}^{\infty} \frac{z^{\nu+2n}}{\Gamma(n + \nu + 1) n!}. \tag{1.3.17}\]

Clearly \(I_{\nu}(z) = e^{-i\pi\nu/2} J_{\nu}(ze^{i\pi/2})\). Furthermore
\[J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z. \tag{1.3.18}\]
The Bessel functions satisfy the recurrence relation
\[ \frac{2\nu}{z} J_{\nu}(z) = J_{\nu+1}(z) + J_{\nu-1}(z). \] (1.3.19)

The Bessel functions \( J_{\nu} \) and \( J_{-\nu} \) satisfy
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0. \] (1.3.20)

When \( \nu \) is not an integer \( J_{\nu} \) and \( J_{-\nu} \) are linear independent solutions of (1.3.20) whose Wronskian is (Watson, 1944, §3.12)
\[ W\{J_{\nu}(x), J_{-\nu}(x)\} = -\frac{2 \sin(\nu\pi)}{\pi x}, \quad W\{f, g\} := fg' - gf'. \] (1.3.21)

The function \( I_{\nu}(x) \) satisfies the differential equation
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0, \] (1.3.22)
whose second solution is
\[ K_{\nu}(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_{\nu}(x)}{\sin(\pi\nu)}, \]
\[ K_n(x) = \lim_{\nu \to n} K_{\nu}(x), \quad n = 0, \pm 1, \ldots. \] (1.3.23)

We also have
\[ I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_{\nu}(x), \]
\[ K_{\nu+1}(x) - K_{\nu-1}(x) = \frac{2\nu}{x} K_{\nu}(x). \] (1.3.24)

**Theorem 1.3.1** When \( \nu > -1 \), the function \( z^{-\nu} J_{\nu}(z) \) has only real and simple zeros. Furthermore, the positive (negative) zeros of \( J_{\nu}(z) \) and \( J_{\nu+1}(z) \) interlace for \( \nu > -1 \).

We shall denote the positive zeros of \( J_{\nu}(z) \) by \( \{j_{\nu,k}\} \), that is
\[ 0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots. \] (1.3.25)

The Bessel functions satisfy the differential recurrence relations, (Watson, 1944)
\[ zJ'_{\nu}(z) = \nu J_{\nu}(z) - zJ_{\nu+1}(z), \] (1.3.26)
\[ zY'_{\nu}(z) = \nu Y_{\nu}(z) - zY_{\nu+1}(z), \] (1.3.27)
\[ zI_{\nu}(z) = zI_{\nu+1}(z) + \nu I_{\nu}(z), \] (1.3.28)
\[ zK'_{\nu}(z) = \nu K_{\nu}(z) - zK_{\nu+1}(z), \] (1.3.29)
where \( Y_{\nu}(z) \) is
\[ Y_{\nu}(z) = \frac{J_{\nu}(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi}, \quad \nu \neq 0, \pm 1, \ldots, \] (1.3.30)
\[ Y_n(z) = \lim_{\nu \to n} Y_{\nu}(z), \quad n = 0, \pm 1, \ldots. \]

The functions \( J_{\nu}(z) \) and \( Y_{\nu}(z) \) are linearly independent solutions of (1.3.20).