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A Cautionary Note on Natural Hedging of Longevity Risk

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In this article, we examine the so-called natural hedging approach for life insurers to internally manage their longevity risk exposure by adjusting their insurance portfolio. In particular, unlike the existing literature, we also consider a nonparametric mortality forecasting model that avoids the assumption that all mortality rates are driven by the same factor(s).

Our primary finding is that higher order variations in mortality rates may considerably affect the performance of natural hedging. More precisely, although results based on a parametric single factor model—in line with the existing literature—imply that almost all longevity risk can be hedged, results are far less encouraging for the nonparametric mortality model. Our finding is supported by robustness tests based on alternative mortality models.

1. INTRODUCTION

Longevity risk, the risk that policyholders will live longer than expected, has recently attracted increasing attention from both academia and insurance practitioners. Different ways have been suggested on how to manage this risk, for example, by transferring it to the financial market via mortality-linked securities (see, e.g., Blake et al. 2006). One approach that is particularly appealing at first glance since it can be arranged from within the insurer is “natural hedging,” that is, adjusting the insurance portfolio to minimize the overall exposure to systematic mortality risk (longevity risk).

Cox and Lin (2007) first formally introduce this concept of mortality risk management for life insurers. They find that, empirically, companies selling both life and annuity policies generally charge cheaper prices for annuities than companies with only a single business line. Since then a number of studies have occurred in the insurance literature showing “that natural hedging can significantly lower the sensitivity of an insurance portfolio with respect to mortality risk” (Gatzert and Wesker 2012, p. 398; see also Bayraktar and Young 2007; Wetzel and Zwiesler 2008; Tsai et al. 2010; Wang et al. 2010).

However, these contributions arrive at their positive appraisal of the natural hedging approach within model-based frameworks; that is, their conclusions rely on conventional mortality models such as the Lee-Carter model (Lee and Carter 1992) or the Cairns-Blake-Dowd (CBD) model (Cairns et al. 2006b). While these popular models allow for a high degree of numerical tractability and serve well for many purposes, they come with the assumption that all mortality rates are driven by the same low-dimensional stochastic factors. Therefore, these models cannot fully capture disparate shifts in mortality rates at different ages, which could have a substantial impact on the actual effectiveness of natural hedging.

To analyze the impact of the mortality forecasting model on the effectiveness of natural hedging, in this article, we compare results under several assumptions for the future evolution of mortality in the context of a stylized life insurer. In particular, aside from considering deterministic mortality rates and a simple-factor model as in previous studies, we also use a nonparametric forecasting model that arises as a by-product of the mortality modeling approach presented in Zhu and Bauer (2013). The advantage of a nonparametric model is that we do not make functional assumptions on the mortality model, especially the potentially critical factor structure indicated above.\textsuperscript{1} Our results reveal that the efficiency of natural hedging is considerably reduced when relying on the nonparametric model—which underscores the problem when relying on model-based analyses for risk management decisions more generally.

We perform various robustness tests for this finding. In particular, we consider a setting without financial risk, and we repeat the calculations for alternative mortality models. Although these analyses reveal additional insights, the primary result is robust.
to these modifications: Natural hedging only marginally reduces the exposure of the company to systematic mortality risk. This meager performance can be viewed as further evidence endorsing market-based solutions for managing longevity risk.

The remainder of the article is structured as follows: Section 2 briefly introduces the considered mortality forecasting models. Section 3 discusses the calculation of the economic capital for a stylized life insurance company, and Section 4 revisits the natural hedging approach within our economic capital framework. Section 5 conducts the robustness tests. Section 6 concludes.

2. MORTALITY FORECASTING MODELS

We commence by introducing the mortality forecasting models that will be primarily used in this article. In particular, we consider two representative models within the forward-mortality framework developed in Zhu and Bauer (2013): a parametric single-factor model and a nonparametric model for the annualized mortality innovations. Employing two models from the same framework facilitates the interpretation of similarities and differences within certain applications. Moreover, as is detailed in Zhu and Bauer (2011), the use of conventional spot mortality models (Cairns et al. 2006a) will typically require so-called nested simulations in the numerical realizations within our Economic Capital framework, which in turn will considerably increase the computational difficulty of the optimization procedures described below. Since we are primarily interested in how the assumption of a low-dimensional factor structure—rather than the choice of any specific mortality forecasting model—affects the performance of the natural hedging approach in model-based analyses, we believe that our model choice serves well as a representative example to draw more general conclusions. Nevertheless, in Section 5 we conduct robustness tests of our results based on alternative factor and nonparametric models that are used in existing literature.

Underlying the approach is a time series of generation life tables for some population for years \( t_1, t_2, \ldots, t_N \). More precisely, in each of these tables labeled by its year \( t_j \), we are given forward-looking survival probabilities \( \tau p_x(t) \) for ages \( x = 0, 1, 2, \ldots, 100 \) and terms \( \tau = 0, 1, 2, \ldots, 101 - x \), where \( \tau p_x(t) \) denotes the probability for an \( x \)-year-old to survive for \( \tau \) periods until time \( t + \tau \). Mathematically this is equivalent to

\[
\tau p_x(t) 1_{\{\Upsilon_{x,t}\leq \tau \}} = E^F[1_{\{\Upsilon_{x,t}\leq \tau + \tau \}} | F_t] \vee \{\Upsilon_{x,t} > \tau \}
\]

for an \((x - t) > 0\) year old at time 0, where \( \Upsilon_{x,t} \) denotes the (random) time of death or future lifetime of an \( x \)-year-old at time 0. In particular, \( \tau p_x(t) \) will account for projected mortality improvements over the future period \([t, t + \tau]\).

Now, for each year \( t_j, 1 \leq j < N \), and for each term/age combination \((\tau, x)\) with \( 1 \leq x \leq 100 \) and \( 0 \leq \tau \leq 100 - x \), we define

\[
F(t_j, t_{j+1}, (\tau, x)) = -\log \left\{ \frac{\tau p_x(t_{j+1})}{\tau p_x(t_j)} \left\{ \tau_{j+1} p_{x-t_{j+1}+t_j}(t_j) \right\} \right\}, 1 \leq j < N.
\]

Hence, \( F(t_j, t_{j+1}, (\tau, x)) \) measures the log-change of the one-year marginal survival probability for an individual aged \( x \) at time \( t_{j+1} \) over the period \([t_{j+1} + \tau, t_j + t + \tau + 1]\) from projection at time \( t_{j+1} \) relative to time \( t_j \). Further, we define the vector \( \tilde{F}(t_j, t_{j+1}) = \text{vec}(F(t_j, t_{j+1}, (\tau, x)), 1 \leq x \leq 100, 0 \leq \tau \leq 100 - x\), with \( \text{dim}(\tilde{F}(t_j, t_{j+1})) = \frac{100 \times 101}{2} = 5,050\), \( j = 1, 2, \ldots, N - 1 \).

Proposition 2.1 in Zhu and Bauer (2013) shows that under the assumption that the mortality age/term structure is driven by a time-homogeneous diffusion and with equidistant evaluation dates, that is, \( t_{j+1} - t_j = \Delta \), \( \tilde{F}(t_j, t_{j+1}), j = 1, \ldots, N - 1 \), are independent and identically distributed (iid). Therefore, in this case a nonparametric mortality forecasting methodology is immediately given by bootstrapping the observations \( \tilde{F}(t_j, t_{j+1}), j = 1, \ldots, N - 1 \) (Efron, 1979). More precisely, with equation (1), we can generate simulations for the generation life tables at time \( t_j+t_{j+1} \) by \{\tilde{F}(t_j, t_{j+1})\}, from \{\tilde{F}(t_j, t_{j+1})\}, \( j = 1, \ldots, N - 1 \) in combination with the known generation life tables at time \( t_j \), \{\tau p_x(t_N)\}. This serves as the algorithm for generating our nonparametric mortality forecasts. A related approach that we consider in the robustness tests (Section 5) relies on the additional assumption that \( \tilde{F}(t_j, t_{j+1}) \) are iid Gaussian random vectors. In this case we can directly sample from a Normal distribution with the mean and the covariance matrix estimated from the sample.

To introduce corresponding factor models, it is possible to simply perform a factor analysis of the iid sample \{\tilde{F}(t_j, t_{j+1})\}, \( j = 1, \ldots, N - 1 \), which shows that for population mortality data, the first factor typically captures the vast part of the systematic variation in mortality forecasts. However, as is detailed in Zhu and Bauer (2013), factor models developed this way are not necessarily self-consistent; that is, expected values derived from simulations of future survival probabilities do not necessarily align with the forecasts engrained in the current generation life table at time \( t_N \).

\footnote{In particular, in this article, we use a maximal age of 101 though generalizations are possible.}
To obtain self-consistent models, it is convenient to introduce the so-called forward force of mortality (Cairns et al. 2006a),

$$\mu_t(\tau, x) = -\frac{\partial}{\partial \tau} \log \{ \tau p_x(t) \},$$

so that we have

$$\tau p_x(t) = \exp \left\{ -\int_0^\tau \mu_s(s, x) ds \right\}. \quad (2)$$

Time-homogeneous (forward) mortality models can then be represented by an infinite-dimensional stochastic differential equation of the form

$$d\mu_t = (A_{\mu} \mu_t + \alpha) dt + \sigma dW_t, \quad \mu_0(\cdot, \cdot) > 0, \quad (3)$$

where $\alpha$ and $\sigma$ are sufficiently regular, function-valued stochastic processes, $A = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x}$, and $(W_t)$ is a $d$-dimensional Brownian motion. Bauer et al. (2012a) show that for self-consistent models, we have the drift condition

$$\alpha(\tau, x) = \sigma(\tau, x) \times \int_0^\tau \sigma'(s, x) ds, \quad (4)$$

and for time-homogeneous, Gaussian models (where $\alpha$ and $\sigma$ are deterministic) to allow for a factor structure, a necessary and sufficient condition is

$$\sigma(\tau, x) = C(x + \tau) \times \exp\{M\tau\} \times N,$$

for some matrices $M, N$, and a vector-valued function $C(\cdot)$. By aligning this semiparametric form with the first factor derived in a factor analysis described above, Zhu and Bauer (2013) propose the following specification for the volatility structure in a single-factor model:

$$\sigma(\tau, x) = \left( k + c e^{d(x+\tau)} \right) (\alpha + \tau) e^{-br}. \quad (5)$$

Together with equations (2), (3), and (4), equation (5) presents the parametric factor mortality forecasting model employed in what follows. We refer to Zhu and Bauer (2013) for further details, particularly on how to obtain maximum-likelihood estimates for the parameters $k, c, d, a$, and $b$.

3. ECONOMIC CAPITAL FOR A STYLIZED INSURER

In this section we employ the mortality forecasting approaches outlined in the previous section to calculate the Economic Capital (EC) of a stylized life insurance company. We start by introducing the framework for the EC calculations akin to Zhu and Bauer (2011). Subsequently we describe the data used in the estimation of the underlying models and resulting parameters. In addition to calculating the EC for a base case company with fixed investments, we derive an optimal static hedge for the financial risk by adjusting the asset weights.

3.1. EC Framework

Consider a (stylized) newly founded life insurance company selling traditional life insurance products only to a fixed population. More specifically, assume that the insurer’s portfolio of policies consists of $n_{x,i}^{\text{term}}$ $i$-year term-life policies with face value $B_{\text{term}}$ for $x$-year old individuals, $n_{x,i}^{\text{end}}$ $i$-year endowment policies with face value $B_{\text{end}}$ for $x$-year old individuals, and $n_{x,i}^{\text{ann}}$ single-premium life annuities with an annual benefit of $B_{\text{ann}}$ paid in arrears for $x$-year old individuals, $x \in X, i \in I$. Furthermore, assume that the benefits/premiums are calculated by the Equivalence Principle based on the concurrent generation table and the concurrent yield curve without the considerations of expenses or profits. In particular, we assume that the insurer is risk-neutral with respect to mortality risk; that is, the valuation measure $\mathbb{Q}$ for insurance liabilities is the product measure of the risk-neutral measure for financial and the physical measure for demographic events.

Under these assumptions, the insurer’s Available Capital at time zero, $AC_0$, defined as the difference of the market value of assets and liabilities, simply amounts to its initial equity capital $E$. The available capital at time 1, $AC_1$, on the other hand, equals
to the difference in the value of the insurer’s assets and liabilities at time 1, denoted by $A_1$ and $V_1$, respectively. More specifically, we have

$$A_1 = \left( E + B_{ann} \sum_{x \in X} a_x(0) n_{x,i}^\text{ann} + B_{term} \sum_{x \in X, i \in I} \frac{A_{x-\tau}(0)}{\delta_{x-\tau}(0)} n_{x,i}^\text{term} + B_{end} \sum_{x \in X, i \in I} \frac{A_{x-\tau}(0)}{\delta_{x-\tau}(0)} n_{x,i}^\text{end} \right) \times R_1,$$

$$V_1 = B_{ann} \sum_{x \in X} \tilde{a}_{x+1}(1) \left( n_{x,i}^\text{ann} - \mathcal{D}_x^\text{ann}(0,1) \right) + B_{term} \sum_{x \in X, i \in I} \mathcal{D}_x^\text{term}(0,1) + B_{end} \sum_{x \in X, i \in I} \mathcal{D}_x^\text{end}(0,1)$$

$$+ B_{term} \sum_{x \in X, i \in I} \left[ A_{x+1\tau}(1) - \frac{A_{x-\tau}(0)}{\delta_{x-\tau}(0)} \tilde{a}_{x+1\tau}(1) \right] \times \left( n_{x,i}^\text{term} - \mathcal{D}_x^\text{term}(0,1) \right)$$

$$+ B_{end} \sum_{x \in X, i \in I} \left[ A_{x+1\tau}(1) - \frac{A_{x-\tau}(0)}{\delta_{x-\tau}(0)} \tilde{a}_{x+1\tau}(1) \right] \times \left( n_{x,i}^\text{end} - \mathcal{D}_x^\text{end}(0,1) \right).$$

Here $R_1$ is the total return on the insurer’s asset portfolio. $\mathcal{D}_x^\text{com}(0,1)$ is the number of deaths between time 0 and time 1 in the cohort of $x$-year-old policyholders with policies of term $i$ and of type con $\in \{\text{ann, term, end}\}$. And $\tilde{a}_x(t)$, $A_{x-\tau}(t)$, etc. denote the present values of the contracts corresponding to the actuarial symbols at time $t$, which are calculated based on the yield curve and the generation table at time $t$. For instance,

$$\tilde{a}_x(t) = \sum_{\tau=0}^{\infty} t^\tau p_x(t) p(t, \tau),$$

where $t^\tau p_x(t)$ is the time-$t$ (forward) survival probability as defined in Section 2, and $p(t, \tau)$ denotes the time $t$ price of a zero-coupon bond that matures in $\tau$ periods (at time $t+\tau$).

The EC calculated within a one-year mark-to-market approach of the insurer can then be derived as (Bauer et al. 2012b)

$$EC = \rho \left( AC_0 - p(0,1) AC_1 \right) \left/ L \right.$$

where $L$ denotes the one-period loss and $\rho(\cdot)$ is a monetary risk measure. For example, if the EC is defined based on Value-at-Risk (VaR) such as the Solvency Capital Requirement (SCR) within the Solvency II framework, we have

$$EC = SCR = \text{VaR}_\alpha(L) = \arg \min_{x} [P(L > x) \leq 1 - \alpha],$$

where $\alpha$ is a given threshold (99.5% in Solvency II). If the EC is defined based on the Conditional Tail Expectation (CTE), on the other hand, we obtain

$$EC = \text{CTE}_\alpha = \mathbb{E}[L|L \geq \text{VaR}_\alpha(L)].$$

In this article, we define the economic capital based on VaR (equation [6]), and choose $\alpha = 95\%$.

### 3.2. Data and Implementation

For estimating the mortality models in this article, we rely on female U.S. population mortality data for the years 1933–2007 as available from the Human Mortality Database. More precisely, we use ages ranging between 0 and 100 years to compile 46 consecutive generation life tables (years $t_1 = 1963$, $t_2 = 1964$, ..., $t_{46} = 2008$) based on Lee-Carter mean projections, with each generated independently from the mortality experience of the previous 30 years. That is, the first table (year $t_1 = 1963$) uses mortality data from years 1933–1962, the second table ($t_2$) uses years 1934–1963, and so forth.

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3Human Mortality Database. University of California, Berkeley (U.S.), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de.

4More precisely, for the estimation of the Lee-Carter parameters, instead of the original approach we use the modified weighted-least-squares algorithm (Wilmoth 1993) and further adjust $\alpha_t$ by fitting a Poisson regression model to the annual number of deaths at each age (Booth et al. 2002).
Having obtained these generation tables \(\{t, p_{ij}(t_j)\}, j = 1, \ldots, N = 46\), we derive the time series of \(\tilde{F}(t_j, t_{j+1}), j = 1, 2, \ldots, 45\), which serve as the underlying sample for our nonparametric forecasting methodology and as the basis for the maximum likelihood parameter estimates of our mortality factor model.\(^5\) In particular, time \(t_N = 2008\) corresponds to time 0, whereas time \(t_{N+1} = 2009\) corresponds to time 1 in our EC framework. Table 1 displays the parameter estimates of the parametric factor model (5).\(^6\)

For the asset side, we assume that the insurer only invests in 5-, 10-, and 20-year U.S. government bonds as well as an equity index (S&P 500) \(S = (S_t)_{t \geq 0}\). For the evolution of the assets, we assume a generalized Black-Scholes model with stochastic interest rates (Vasicek model), that is, under \(\mathbb{P}\),

\[
dS_t = S_t(\mu dt + \rho \sigma_A dB^{(1)}_t) + \sqrt{1 - \rho^2} \sigma_r dB^{(2)}_t, \quad S_0 > 0,
\]

\[
dr_t = \kappa (\gamma - r_t) dt + \sigma_r dB^{(1)}_t, \quad r_0 > 0,
\]

where \(\mu, \sigma_A, \kappa, \gamma, \sigma_r > 0\) and \(\rho \in [-1, 1]\), and \((B^{(1)}_t)\) and \((B^{(2)}_t)\) are independent Brownian motions that are independent of \((W_t)\). Moreover, we assume that the market price of interest rate risk is constant and denote it by \(\lambda\); that is, we replace \(\mu\) by \(r_t\) and \(\gamma\) by \(\gamma - (\lambda \sigma_r)/\kappa\) for the dynamics under the risk-neutral measure \(\mathbb{Q}\).

We estimate the parameters based on U.S. data from June 1988 to June 2008 using a Kalman filter. In particular, we use monthly data of the S&P 500 index,\(^7\) Treasury bills (three months), and government bonds with maturities of one, three, five, and ten years.\(^8\) The parameter estimates are displayed in Table 2.

Based on time 1 realizations of the asset process, \(S_1\), and the instantaneous risk-free rate, \(r_1\), we have

\[
R_1 = \omega_1 \frac{S_1}{S_0} + \omega_2 \frac{p(1, 4)}{p(0, 5)} + \omega_3 \frac{p(1, 9)}{p(0, 10)} + \omega_4 \frac{p(1, 19)}{p(0, 20)},
\]

where \(\omega_i, i = 1, \ldots, 4\), are the company’s proportions of assets invested in each category. A procedure to generate realizations of \(S_1, r_1\), and \(p(t, \tau)\) with the use of Monte Carlo simulations is outlined in Zaglauer and Bauer (2008).

### 3.3. Results

Table 3 displays the portfolio of policies for our stylized insurer. For simplicity and without loss of generality, we assume that the company holds an equal number of term/endowment/annuity contracts for different age/term combinations and that the face values coincide; of course, generalizations are possible. The initial capital level is set to \(E = 20,000,000\). The insurer’s assets and liabilities at time 0, \(A_0\) and \(V_0\), are calculated at \(\$1,124,603,545\) and \(\$1,104,603,545\), respectively.

---

\(^5\)Of course, the underlying sample of 45 realizations is rather small for generating a large bootstrap sample, which limits the scope of the approach for certain applications (such as estimating VaR for high confidence levels, which is of practical interest). We come back to this point in our robustness tests (Section 5).

\(^6\)A Principal Component Analysis indicates that 85% of the total variation in the \(\tilde{F}(t_j, t_{j+1}), j = 1, 2, \ldots, 45\), is explained by the leading factor for our dataset. Generally, the percentage of total variation explained is slightly larger for female data in comparison to male data (Zhu and Bauer 2013), suggesting that for female populations a single-factor model is more appropriate.


\(^8\)Federal Reserve Economic Data (FRED), http://research.stlouisfed.org/fred2/.
A CAUTIONARY NOTE ON NATURAL HEDGING OF LONGEVITY RISK

TABLE 3
Portfolio of Policies for Stylized Life Insurer

<table>
<thead>
<tr>
<th>x</th>
<th>i</th>
<th>$n_{x,i}$</th>
<th>$B_{x,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>20</td>
<td>2,500</td>
<td>$100,000</td>
</tr>
<tr>
<td>35</td>
<td>15</td>
<td>2,500</td>
<td>$100,000</td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>2,500</td>
<td>$100,000</td>
</tr>
<tr>
<td>45</td>
<td>5</td>
<td>2,500</td>
<td>$100,000</td>
</tr>
</tbody>
</table>

Term life

Endowment

Annuities

We consider three different approaches to modeling mortality risk: (1) a deterministic evolution of mortality given by the life table at time 0 (2008), \{t, p_x(t)\}; (2) the parametric factor model (5); and, (3), the nonparametric mortality model also introduced in Section 2. Within each approach, we use 50,000 simulations of the assets and liabilities to generate realizations of the loss (L) at time 1, where in addition to financial and systematic mortality risk, we also consider unsystematic mortality risk by sampling the number of deaths within each cohort. Finally, we can calculate the EC via the resulting empirical distribution functions and the given risk measure $\rho$. In particular, for VaR we rely on the empirical quantile. Table 4 displays the results for two assumptions regarding the insurer’s investments.

For the results in the first row of Table 4, we assume that the company does not optimize its asset allocation but invests a fixed 30% in the equity index (see, e.g., ACLI 2011), and the rest in government bonds to match the duration of its liabilities (at 10.2560). Without stochastic mortality, we find EC levels of around $60,000,000, which suggests that the current capital position of $20,000,000 is not sufficient; that is, the firm is undercapitalized. Surprisingly, including systematic mortality risk appears to have little influence on the results in this case: The EC increases by only $787,832 (1.30%) or $2,004,332 (3.30%) when introducing mortality risk via the factor mortality model or nonparametric mortality model, respectively.

However, this changes dramatically when we allow the insurer to pursue a more refined allocation strategy to better manage the financial risk exposure. In the second row of Table 4, we display the results when the insurer optimally chooses (static) asset weights in order to minimize the EC. The corresponding portfolio weights are displayed in Table 5. We find that although the EC level decreases vastly under all three mortality assumptions so that the company is solvent according to the 95% VaR capital requirement ($EC \leq AC_0$), the relative impact of systematic mortality risk now is highly significant. More precisely, the (minimized) EC increases by 208.31% (to $9,871,987) and 213.86% (to $10,049,401) if systematic mortality risk is considered via the factor model and nonparametric model, respectively. This underscores an important point in the debate about the economic relevance of mortality and longevity risk: Although financial risk indices may be more volatile and thus may dominate systematic mortality risk, there exist conventional methods and (financial) instruments to hedge against financial risk.

Of course, naturally the question arises if we can use a similar approach to protect against systematic mortality risk, either by expanding the scope of securities considered on the asset side toward mortality-linked securities or by adjusting the composition of the insurance portfolio on the liability side. The former approach has been considered in a number of papers (see, e.g., Cairns et al. 2013, Li and Luo 2012; and references therein), but a liquid market of corresponding instruments is only slowly emerging. The latter approach—which is commonly referred to as natural hedging (Cox and Lin 2007) and which is in the focus of this

TABLE 4
Economic Capital for Different Investment Strategies

<table>
<thead>
<tr>
<th></th>
<th>Deterministic Mortality</th>
<th>Factor Model</th>
<th>Nonparametric Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>95% VaR (no hedging)</td>
<td>$60,797,835</td>
<td>$61,585,667</td>
<td>$62,802,167</td>
</tr>
<tr>
<td>95% VaR (financial hedging)</td>
<td>$3,201,921</td>
<td>$9,871,987</td>
<td>$10,049,401</td>
</tr>
</tbody>
</table>
We start by considering the factor mortality model and compare it to the case without stochastic mortality risk. Figure 1 shows the EC as a function of the number of term-life policies in the insurer’s portfolio \(n^{\text{term}}\) (we will refer to this as an “EC curve”). We first find that in the case of no systematic mortality risk (deterministic mortality), EC increases in the number of term policies. The reason is twofold: On the one hand, an increase leads to higher premiums and, thus, assets, which increases asset risk. On the other hand, although there is no systematic mortality risk, the number of deaths in each cohort is a random variable due to nonsystematic mortality risk, which clearly increases in the number of policies. In contrast, under the stochastic factor mortality model, the EC is a convex function of \(n^{\text{term}}\), which initially decreases and then increases sharply; that is, it is U-shaped. The optimal number of policies, \(n^{\text{term}}^*\), is approximately 60,000 and the corresponding minimal EC is $4,165,973, which is only slightly larger than the corresponding EC level under deterministic mortality ($4,128,345). Therefore, in line with other papers on natural hedging, it appears that an appropriately composed insurance portfolio can serve well for hedging against systematic mortality/longevity risk.

However, when repeating the same exercise based on the nonparametric forecasting model, the situation changes considerably. As is also depicted in Figure 1, in this case we can observe an only very mild effect of natural hedging when \(n^{\text{term}}\) is relatively small, and it is far less pronounced compared with the factor model. In particular, at the optimal term-insurance exposure \(n^{\text{term}}^* = 60,000\), the capital level is at $13,872,739 for the nonparametric model, which is far greater than the corresponding capital level under deterministic mortality ($4,128,345).

The intuition for this result is as follows: As indicated at the end of Section 3.3, the two models behave quantitatively alike in “normal” circumstances and particularly yield similar capital levels for the initial portfolio. This is not surprising since the rationale behind the single factor model—akin to other single-factor models such as the Lee-Carter model—is that the majority of the variation across ages and terms can be explained by the leading factor (85% for our dataset). Essentially, the residuals for

\[ \text{Deterministic Mortality} \quad \text{Factor Model} \quad \text{Nonparametric Model} \]

<table>
<thead>
<tr>
<th>Stock</th>
<th>0.2%</th>
<th>1.5%</th>
<th>0.9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-year bond</td>
<td>2.5%</td>
<td>0.1%</td>
<td>0.5%</td>
</tr>
<tr>
<td>10-year bond</td>
<td>87.3%</td>
<td>88.0%</td>
<td>90.8%</td>
</tr>
<tr>
<td>20-year bond</td>
<td>10.0%</td>
<td>10.4%</td>
<td>7.8%</td>
</tr>
</tbody>
</table>
lower ages are small in absolute terms and mostly unsystematic, whereas the residuals for higher ages (beyond 50) are relatively large in absolute terms and mostly systematic. The latter are responsible for the high proportions of the variation explained in absolute terms. However, under the natural hedging approach, the large exposure in the term-life lines leads to a considerable rescaling of the profile of the residuals across terms and ages, so that this similarity breaks down. In particular, the residuals for lower-age groups become increasingly important, which in turn are considerably influenced by higher-order factors including but also beyond the second factor—some of which do not carry a systematic shape at all. Thus, for the analysis of the effectiveness of natural hedging, the consideration of higher order/nonsystematic variation indeed might be important.

Again, we would like to emphasize that this is not general criticism of these models. For many applications, such as forecasting mortality rates, abstracting from these small and unsystematic variations is expedient. We solely challenge the reliance on low-dimensional factor models for the analysis of hedging performance. Indeed, our results indicate that natural hedging may not be as effective as asserted in the existing literature.

5. ROBUSTNESS OF THE RESULTS

Of course, the question may arise to what extent the results on the performance of natural hedging are driven by the details of our setup. Thus, in this section we repeat the EC calculations under modified assumptions. In particular, we examine the impact of financial risk on the results, we consider modifications of our mortality models, and we derive EC curves for alternative mortality models.

5.1. The Impact of Financial Risk

To analyze the role played by financial risk in the results, we recalculate the EC levels for different term-life exposures under a deterministic evolution of the asset side—so that the results are solely driven by systematic and unsystematic mortality risk. More precisely, in our asset model (equation [8]), we set both volatility terms $\sigma_A$ and $\sigma_r$ to zero, and we use the risk-neutral drift parameters throughout. In particular, the equity index $S$ is now risk-less and returns the risk-free rate. Figure 2(a) displays the corresponding optimal EC curves.
We find that the EC levels under each of the mortality models are similar to the case with financial risk (Fig. 1). For large values of \( n_{\text{term}} \), the EC here even exceeds the corresponding values in the case with financial risk, which appears surprising at first glance. The reason for this observation is the change of probability measure in the financial setting and the associated risk premia paid over the first year. For comparison, we also plot the EC curves for all mortality assumptions when relying on the \( \mathbb{Q} \)-measure throughout in Figure 2(b). We find that the (hypothetical) EC, as expected, is always greater than without the consideration of financial risk, though the difference is not very pronounced. This indicates that the static hedging procedure eliminates most of the financial risk or, in other words, that financial risk does not contribute too much to the total EC.

5.2. Modifications of the Mortality Models

As indicated in note 5, the relatively small size of the sample underlying our nonparametric forecasting approach may be problematic for certain applications such as estimating VaR for a high confidence level. To analyze the impact of the small sample size on our results, we follow the approach also described in Section 2 that relies on the additional assumption that \( F(t_N, t_{N+1}) \) is Gaussian distributed. Then, rather than sampling \( F(t_N, t_{N+1}) \) from the empirical realizations, we generate random vectors with the mean vector and the covariance matrix estimated from the underlying sample. Figure 3(a) shows the resulting EC curve in comparison to the deterministic mortality case. We find that the results are very similar to the nonparametric model underlying Figure 1. In particular, there is only a rather mild effect of natural hedging when \( n_{\text{term}} \) is small, and the economic capital levels considerably exceed those for the deterministic mortality case.

As also was indicated in Section 2, the competing model used in the calculations in Sections 3 and 4, while originating from a factor analysis, presents a self-consistent, parametric approximation. In particular, the entire term structure is driven by only a handful parameters in this case, so that it is not immediately clear what aspects of the model are responsible for the results. Thus, as an intermediary step, we also provide results based on a (high-dimensional) single-factor model. In particular, instead of directly relying on the leading principal component, we estimate a one-factor model following the approach from Bai and Li (2012), which allows for heteroscedasticity in the error term so that some variation will also be picked up for lower ages. More precisely, this approach posits a factor form

\[
F(t_j, t_{j+1}) = \alpha + \beta \star \lambda_t + \delta_t, \ j = 1, \ldots, N - 1,
\]

where \( \alpha \) and \( \beta \) are 5050 \times 1 vectors, \( \mathbb{E}[\delta_t] = 0 \) and \( \mathbb{E}[\delta_t \times \delta_t'] = \Sigma_{\text{error}} = \text{diag}(\sigma_1^2, \ldots, \sigma_{5050}^2) \), which is estimated via maximum likelihood. Here, we employ the leading factor from the Principal Component Analysis as the starting value in the numerical optimization of the log-likelihood, and the resulting factors overall are very similar.
Figure 3(b) provides the EC curve based on this factor model. We find that the effect of natural hedging is far less pronounced than for the parametric factor model, and the Economic Capitals are considerably higher than for the deterministic mortality case throughout. However, we also observe that in contrast to the nonparametric approach in Figure 1, the EC curve is “flat” in the sense that the increased exposure to term-life insurance has only a little effect on the economic capital level. This indicates that the factor loadings for the lower age range are very small, or, in other words, that much of the variation is driven by higher order variations. For the parametric factor model, on the other hand, the parametric form is fit across all terms and ages, which appears to yield a more significant relationship between low and high ages in the first factor. Thus, the parametric nature of the model also seems to be an important driver for the positive appraisal of the natural hedging approach, at least in our setting.

5.3. Alternative Mortality Models

As a final robustness check, we repeat the calculations for alternative mortality models that do not fall within our framework. We start by providing results for the (stochastic) Lee-Carter model as another model where all variation is driven by a single factor. More precisely, we use the Lee-Carter parameters estimated at time \( t_N \)—which also serve for generating the corresponding generation life table that we use for the calculation of time 0 premiums—and simulations of the \( \kappa_{t+1} \) to generate life tables at time 1 with each based on the median projection starting from a simulated value of \( \kappa_{t+1} \). Figure 4(a) presents the resulting EC curve.

We make two primary observations. On the one hand, the EC curve exhibits a U-shape similar to the parametric factor model in Figure 1, that is, natural hedging again is found to be highly effective under this model. In particular, the optimal exposure to term life policies \( n_{\text{term}} \) again is around 60,000 with a corresponding minimum capital of $4,150,010. This finding is not surprising since it was exactly this positive appraisal of the natural hedging approach in previous contributions that serves as the primary motivation of this article. On the other hand, we observe that the magnitude of EC is considerably lower than in the models considered in Sections 3 and 4. Again, this finding is not surprising since it is exactly the underestimation of the risk in long-term mortality trends that serves as the motivation for the underlying approach in Zhu and Bauer (2013).

As a second nonparametric modeling approach, we implement the model proposed by Li and Ng (2010), which relies on bootstrapping one-year mortality reduction factors \( r_{x,t} = \frac{m_{x+1,t}}{m_{x,t}} \) for \( j = 1, \ldots, N - 1 \) and \( m_{x,t} \) being the central death rate for age \( x \) in year \( t \). More precisely, as proposed in Li and Ng (2010), we use a block bootstrap method with a block size of two to capture the serial dependency in the data. For consistency with the other mortality models in this article, we use 30 years of the historical data (1978–2007) and ages \( x \) ranging from 0 to 100. Figure 4(b) shows the resulting EC curves, where, of course, the deterministic curve is calculated based on a generation table compiled also using this model.

The capital levels are lower than for the approaches considered in Sections 3 and 4 though larger than for the (stochastic) Lee-Carter model, which is due to differences in the model structures. However, the observations regarding natural hedging are in
Exposure of term-life
Optimal economic capital
deterministic mortality
stochastic model

\( (a) \) Lee-Carter model

\( (b) \) One-factor nonparametric model

FIGURE 4. Alternative Mortality Models. (color figure available online)

line with our results from Section 4. More precisely, the effect is not very pronounced under the nonparametric model—we see only a very mild U-shape—and the EC level increases considerably for higher values of \( n_{\text{term}} \).

6. CONCLUSION

In this article, we analyze the effectiveness of natural hedging in the context of a stylized life insurer. Our primary finding is that higher order variations in mortality rates may considerably affect the performance of natural hedging. More precisely, although results based on a parametric single-factor model imply that almost all longevity risk can be hedged by appropriately adjusting the insurance portfolio (in line with the existing literature), the results are far less encouraging when including higher order variations via a nonparametric mortality forecasting model.

Of course, this is not a general endorsement of these more complicated models. Simple (or parsimonious) models may have many benefits in view of their tractability, their statistical properties, or their forecasting power. We solely show that relying on “simple” models for analyzing the performance of hedges may be misleading since they contain assumptions on the dependence across ages that are not necessarily supported by the data.

At a broader level, we believe our results call for more caution toward model-based results in the actuarial literature in general.

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REFERENCES


*Discussions on this article can be submitted until October 1, 2014. The authors reserve the right to reply to any discussion. Please see the Instructions for Authors found online at http://www.tandfonline.com/uaaj for submission instructions.*
A closed-form approximation for valuing European basket warrants under credit risk and interest rate risk

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Over the past few years, many financial institutions have actively traded basket warrants in the over-the-counter market. Prior research has proposed an approach to valuing single-stock options subject to credit. However, this approach cannot be applied directly to the case of basket warrants. Using the martingale method, we propose a closed-form approximation for valuing European basket warrants using a continuous-time model, with credit risk and interest rate risk considered simultaneously. Finally, several numerical examples are utilized to demonstrate the characteristics of basket warrants under credit risk.

Keywords: Derivatives pricing; Derivatives securities; Stochastic interest rates; Credit risk

JEL Classification: G1, G13

1. Introduction

With the liberalization of global financial markets and the instability of the world economy, many derivatives have been developed to meet the increasing needs of investors. Among these derivative securities, basket warrants have gradually become more popular over the past decade. In essence, basket warrants are actually basket options. These options have a basket of two or more underlying assets whose prices determine basket warrants’ payoffs. However, basket warrants and ordinary options are different in terms of issuing institutions. Basket warrants are normally issued by financial institutions such as investment banks. Investors thus face the credit risk of issuers. As the global financial markets are rapidly changing, investors are also concerned with interest rate risk. The purpose of this paper is to value European basket warrants, with credit risk and interest rate risk considered simultaneously.

Since securities companies which issue basket warrants may default on their obligations, investors should take into account the creditworthiness of issuing organizations when purchasing warrants. Johnson and Stulz (1987) was one of the first studies to examine the pricing of these options with default risk, also known as vulnerable options. Hull and White (1995), Jarrow and Turnbull (1995), Klein (1996), and Hung and Liu (2005) also indicate the importance of taking into account counterparty risk when pricing options traded on the over-the-counter (OTC) market due to the unavailability of a clearing mechanism.

When valuing vulnerable options, Klein (1996) does not consider the interest rate risks, while Johnson and Stulz (1987) assume a fixed interest rate. Both Gentle (1993) and Milevsky and Posner (1998) also assume fixed interest rates when valuing basket options. Based on the assumption of fixed interest rates, the value of warrants would be underestimated if interest rates are highly volatile before maturity.

Hull and White (1995) and Jarrow and Turnbull (1995) assume independence between the assets of the option writer and the underlying asset of the option, especially when the option writer is a large and well diversified financial institution (Hull and White 1995). However, Klein (1996) argues that the option writer may still default due to the volatile changes in the value of the underlying asset even if the option writer has undertaken hedging. Johnson and Stulz

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†It is noted that Gentle (1993) and Milevsky and Posner (1998) do not consider default risk when pricing basket options.

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(1987) further suggest that the option writer may default, possibly due to decreases in the value of the assets of the writer and/or the growth of the value of the option. If the value of the option grows to a larger extent than that of the assets of the writer, it is likely that default may occur.

Prior studies also discuss the correlations among the assets of the option writer, the assets underlying the options, and the interest rate. Merton (1974) was one of the first to study the link between the value of the assets of the option writer and the default event, considering the relation between the interest rate and the assets underlying the options. Extending Merton’s (1974) work, Longstaff and Schwartz (1995) simultaneously consider both default risk and interest rate risk. They further point out that the changes in the value of the assets of the firm and in the interest rates have a significant impact on credit spread when pricing risky bonds. However, the interest rate risk is generally not taken into account in the literature when valuing vulnerable options.

In this paper, we argue the importance of considering both risks and simultaneously model these two risks when pricing basket warrants for the following reason. In the past few years, warrant writers have been exposed to bankruptcy risk due to the Financial Crisis of 2007–2010. They are also adversely affected by interest rate volatility. Furthermore, prices of financial assets are significantly correlated. Unlike prior studies, we therefore simultaneously take into account credit risk, interest rate risk, and correlations between assets when valuing basket warrants. Our valuation formula provides flexibility in pricing warrants with underlying assets including bonds, stocks and other types of securities. It is actually the general closed-form solution of the models of Black and Scholes (1973), Smith (1976), Gentle (1993), Hull and White (1995), Klein (1996), and Klein and Inglis (1999).

The remainder of the paper is organized as follows. The following section develops a theoretical model for valuing European basket warrants subject to interest rate risk and to financial distress on the part of the warrant writer. Next, we derive a closed-form approximation valuation formula for vulnerable basket call and put warrants. In the penultimate section, we utilize several numerical examples to show the properties of our pricing formula. Finally, the last section concludes this paper.

2. The model

This article’s framework for valuing basket warrants can be considered an extension of the framework for pricing vulnerable options proposed by Klein and Inglis (1999). In their paper, Klein and Inglis (1999) value calls/puts on a single asset, while in this study we price basket warrants. The underlying of basket warrants is a basket of assets. Specifically, the basket of assets is actually a portfolio of \( n \) kinds of different tradable stocks. We assume that the warrants are traded under a continuous-time frame and that markets are perfect and frictionless, i.e. transaction costs and taxes are ignored. Suppose that the market value of each of these \( n \) stocks follows Geometric Brownian Motion (GBM). The dynamics of the market value of a particular stock are then stated as follows:

\[
\frac{dS_i}{S_i} = \mu_S \, dt + \sigma_S \, dW_S, \quad i = 1, \ldots, n, \tag{1}
\]

where \( \mu_S \) and \( \sigma_S \) are the instantaneous expected return on stock \( i \) underlying the warrant and the instantaneous standard deviation of the return (both assumed to be constants), respectively, \( S_0 \) is the price of the \( i \)th stock at time \( t \), \( W_S \) is a standard Wiener process, \( i = 1, \ldots, n \), and \( <W_S, \, W_S> = \rho_{ij} t \), for every \( i, j = 1, \ldots, n \).

Credit risk, also known as default risk, is defined as the risk that warrant writers, such as investment banks or securities firms, will be unable to make the required payments when their debt obligations fall due. The bankruptcy event occurs when the writer’s value of assets at the expiration date \( (V_T) \) are smaller than its value of debts \( (D) \). If the warrant writer is bankrupt, its payments to investors depend on the value of its assets on the warrant’s expiration date. If the value of the assets of the warrant writer \( V_T \) falls below the fixed threshold value \( D^* \) on the warrant maturity date \( T \), then default occurs. The value of \( D^* \) is allowed to be less than the value of \( D \), the outstanding liabilities of the writer. As in Klein and Inglis (1999), \( D \) is simplified as the value of the zero-coupon bonds issued by the writer. We assume that \( D \) is the same as \( D^* \) in the process of valuation to allow for capital forbearance of the warrant writer. Once the writer goes bankrupt, the warrant holder can only claim \((1 - \alpha)V_T/D\), where \( 0 \leq \alpha \leq 1 \) is the costs associated with the financial distress when the writer becomes bankrupt. Suppose that the value of the assets of the warrant writer \( V \) follows a GBM. The dynamic process of \( V \) is as follows:

\[
\frac{dV}{V} = \mu_V \, dt + \sigma_V \, dW_V, \tag{2}
\]

where \( \mu_V \) and \( \sigma_V \) are the instantaneous expected return on the assets of the writer and the instantaneous standard deviation of the return (both assumed to be constants), respectively, and \( W_V \) is a standard Wiener process.

As for the interest rate risk, let \( P(t, T) \) represent the price of a zero-coupon bond at time \( t \) paying one dollar at time \( T \), where \( T \) represents the expiration date of the warrants. Therefore,

\[
\frac{dP(t, T)}{P(t, T)} = \mu_P \, dt + \sigma_P(T - t) \, dW_P, \tag{3}
\]

where \( \mu_P \) and \( \sigma_P \) denote the instantaneous expected return on the zero-coupon bond and the instantaneous standard deviation of the return, respectively, and \( W_P \) follows a standard Wiener process.

Let \( B(t) \) denote a money market account which corresponds to the future value of the wealth accumulated from an investment of \$1 at an interest rate of \( r(t) \). Its dynamic process is \( dB(t) = r(t)B(t) \, dt \), where \( r(t) \) is the instantaneous interest rate at time \( t \).

Under the risk-neutral probability measure \( Q \), the dynamics of the zero coupon bond price are

\[
\frac{dP(t, T)}{P(t, T)} = r(t) \, dt + \sigma_P(T - t) \, dW_P^Q. \tag{4}
\]
As shown in equations (3) and (4), the instantaneous expected return on the zero-coupon bond under the probability measure \( P \) is replaced by the instantaneous interest rate \( r(t) \) under the risk-neutral measure \( \mathcal{Q} \).

In addition, the market value of the underlying stock, the value of the assets of the warrant writer, and the price of the zero-coupon bond are all assumed to be correlated with each other under the probability measure \( P \). The instantaneous correlations between \( W_i \) and \( W_p \), between \( W_s \) and \( W_p \), and between \( W_i \) and \( W_s \) are \( \rho_{WP}, \rho_{WS}, \) and \( \rho_{WS} \), respectively.

Our framework described in this section can be applied to the models proposed by Vasicek (1977), Hull and White (1990), and Heath et al. (1992). It is also worth noting that the valuation formula for the vulnerable European basket warrants is the general closed-form solution of the models of Black and Scholes (1973), Smith (1976), Gentle (1993), Hull and White (1995), Klein (1996), and Klein and Inglis (1999).

In the ‘Numerical examples’ section below, we use the Cox, Ingersoll and Ross (1985) model (CIR model), which is a special case of the Hull and White (1990) model.

### 3. Valuation of vulnerable basket warrants

The warrant writer is considered to be bankrupt if the value of its asset \( V_T \) falls below the value \( D \) on the maturity date. If the writer defaults on its obligations at maturity, the claims of the warrant purchasers would not be completely satisfied. Thus, we use the maturity date \( T \) as a reference point in time.

Let \( C^g_T \) be the payoff of a European basket call warrant at maturity date \( T \). \( C^g_T \) is defined as

\[
C^g_T = \max \{ BK_T - K, 0 \},
\]

where \( BK_T = \sum_{i=1}^{n} w_i S_{iT} \), \( i = 1, \ldots, n \). \( BK_T \) represents the weighted average value of \( n \) kinds of different stocks at time \( T \). \( K \) is the strike price of the basket call warrant, \( w_i \) represents the weight of the \( i \)-th stock and \( \sum_{i=1}^{n} w_i = 1 \). Let \( F^S_t(t, T) = S_0/P(t, T) \), and \( F^S_t(t, T) = \) the forward price at time \( t \) of the \( i \)-th asset for the settlement date \( T \).

\[
BK_T = \sum_{i=1}^{n} w_i S_{iT} = \sum_{i=1}^{n} w_i F^S_t(t, T) \frac{S_{iT}}{F^S_t(t, T)}.
\]

The payoff at maturity \( T \) of a basket call option can be rewritten as follows:

\[
C^g_T = \max \{ BK_T - K, 0 \} = \max \left\{ \sum_{i=1}^{n} \left[ w_i F^S_t(t, T) \frac{S_{iT}}{F^S_t(t, T)} - \frac{K}{\sum_{i=1}^{n} w_i F^S_t(t, T)} \left( \sum_{i=1}^{n} w_i F^S_t(t, T) \right) \right], 0 \right\}
\]

\[
= \sum_{i=1}^{n} w_i F^S_t(t, T) \left\{ \sum_{i=1}^{n} X_i S_{iT} - K^+, \ 0 \right\},
\]

where the modified weight is

\[
X_i = \frac{w_i F^S_t(t, T)}{\sum_{i=1}^{n} w_i F^S_t(t, T)} \quad \sum_{i=1}^{n} X_i = 1,
\]

and

\[
K^+ = \frac{K}{\sum_{i=1}^{n} w_i F^S_t(t, T)}, \quad \hat{S}_T = \frac{S_{iT}}{F^S_t(t, T)}.
\]

If the money market account \( B(t) \) is used as the numeraire, the discounted price of the asset is \( \mathcal{Q} \)-martingale. If \( P(t, T) \), the price of a zero-coupon bond with a maturity date \( T \) at time \( t \), is used as the numeraire, the discounted price of the asset will be \( \mathcal{Q}^D \)-martingale. Since \( P(t, T) = B(T) = 1 \), \( C_{T}^g \), the payoff function of a vulnerable European basket call warrant with a maturity date \( T \), can be given as follows:

\[
C^g_T = \begin{cases} (BK_T - K), & \text{if } BK_T - K > 0 \text{ and } V_T \geq D, \\ (BK_T - K)(1 - \alpha) \frac{V_T}{D}, & \text{if } BK_T - K > 0 \text{ and } V_T < D, \\ 0, & \text{if } BK_T - K \leq 0. \end{cases}
\]

where \( BK_T \) represents the weighted average value of \( n \) kinds of different stocks at time \( T \), \( K \) is the strike price of the basket call warrant, and \( V_T \) and \( D \) are the warrant writer’s assets and liabilities, respectively. When \( V_T < D \), the writer is liquidated and its residual firm value is \( (1 - \alpha)V_T \), where \( \alpha \) is the percentage representing the deadweight costs associated with financial distress and \( 0 \leq \alpha \leq 1 \). The deadweight costs include the direct and indirect costs of bankruptcy. Warrant holders can only claim back \( (1 - \alpha)V_T/D \), where \( (1 - \alpha) \) is the recovery rate. The actual payoff of a European basket call warrant with a mature date \( T \) is then \( (BK_T - K)(1 - \alpha)V_T/D \).

If \( P(t, T) \) is used as the numeraire, \( C^g_T/P(t, T) \) will be the martingale of the probability measure \( \mathcal{Q}^T \). Thus,

\[
E_{\mathcal{Q}^T}\left[ C^g_T \left( \begin{array}{c} 1_{\{V_T \geq D\}} + \frac{(1 - \alpha)V_T}{D} 1_{\{V_T < D\}} \end{array} \right) \right] = C^g_T P(t, T).
\]

The value of a vulnerable European basket call warrant at time \( t \), \( C^g_t \), is

\[
C^g_t = P(t, T)E_{\mathcal{Q}^T}\left[ C^g_T \left( \begin{array}{c} 1_{\{V_T \geq D\}} + \frac{(1 - \alpha)V_T}{D} 1_{\{V_T < D\}} \end{array} \right) \right] F_t.
\]

Equation (6) shows that the expected future payout on the nominal claim of amount \( C^g_T \) depends on the terminal value of the writer’s assets. The nominal claim is paid out in full if the value of the assets of the option writer at maturity \( T \), \( V_T \), is greater than the value of the debt of the writer \( D \). \( E_{\mathcal{Q}^T} \) is the forward risk-neutral expectation of \( \mathcal{Q}^T \).

Substituting equation (5) into (6) gives

\[
C^g_t = P(t, T)E_{\mathcal{Q}^T}\left[ \sum_{i=1}^{n} X_i S_{iT} - K^+ \right] \left( \begin{array}{c} 1_{\{V_T \geq D\}} + \frac{(1 - \alpha)V_T}{D} 1_{\{V_T < D\}} \end{array} \right) F_t \sum_{i=1}^{n} w_i F^S_t(t, T).
\]
Although individual stock prices follow a log-normal distribution, the weighted average price of stocks $B_{Kt}$ is no longer log-normally distributed. Using the fact that an arithmetic average is always greater than a geometric average, Vorst (1992) proposes†

$$X_i \approx \sum_{t=1}^{n} X_i \bar{S}_{it} - E_{Q^t} \left[ \prod_{t=1}^{n} \bar{S}_{it}^m \right] + E_{Q^t} \left[ \sum_{t=1}^{n} X_i \bar{S}_{it} \right]$$

(8)

Substituting equation (8) into (7), the value of a vulnerable European call $C_{S}^v$ can be rewritten as

$$C_{v}^S \approx P(t, T)E_{Q^T} \left[ \left( \sum_{t=1}^{n} S_{i}^v \right)^{-\frac{1}{2}} \left( 1_{\{\bar{V}_r \geq D\}} \right) \right] + \frac{(1 - \delta) \bar{V}_r}{D} E_{Q^T} \left[ \sum_{t=1}^{n} w_i F_{i}^v(t, T) \right] = B_{Kt}[A_1 - A_2 + A_3 - A_4],$$

where

$$A_1 = E_{Q^T} \left[ \left( \sum_{t=1}^{n} S_{i}^v \right)^{-\frac{1}{2}} \left( 1_{\{\bar{V}_r \geq D\}} \right) \right],$$

$$A_2 = E_{Q^T} \left[ K \left( \sum_{t=1}^{n} \bar{S}_{it}^{mS} \right)^{1_{\{\bar{V}_r < D\}}} \right],$$

$$A_3 = E_{Q^T} \left[ \left( \sum_{t=1}^{n} S_{i}^v \right)^{-\frac{1}{2}} \left( \frac{(1 - \delta) \bar{V}_r}{D} \right) \right],$$

$$A_4 = E_{Q^T} \left[ K \left( \sum_{t=1}^{n} \bar{S}_{it}^{mS} \right)^{1_{\{\bar{V}_r < D\}}} \right].$$

$$\delta_{S}(T-t) = \delta_{Sv}(T-t)\delta_{S}(T-t)$$

(9)

$$\gamma = \frac{S_{SV}}{S_{S}},$$

$$a_1 = \frac{\ln(m^2_\delta / K^*) + S_{S}^2}{S_{S}},$$

$$a_2 = \frac{\ln(V_1 / P(t, T)D) - (S_{S}^2 / 2) + S_{SV}}{S_{S}},$$

$$b_1 = \frac{\ln(m^2_\delta / K^*) + S_{S}^2 + S_{SV}}{S_{S}},$$

$$b_2 = -\frac{\ln(V_1 / P(t, T)D) + (S_{S}^2 / 2) + S_{SV}}{S_{S}},$$

$$c_1 = \frac{\ln(m^2_\delta / K^*) + S_{S}^2 + S_{SV}}{S_{S}},$$

$$c_2 = -\frac{\ln(V_1 / P(t, T)D) + (S_{S}^2 / 2) + S_{SV}}{S_{S}},$$

$$d_1 = \frac{\ln(m^2_\delta / K^*) + S_{SV}}{S_{S}},$$

$$d_2 = -\frac{\ln(V_1 / P(t, T)D) + (S_{S}^2 / 2)}{S_{S}}.$$
European options subject to terms of the value of the zero-coupon bond, respectively. Let $PB = \frac{C_2}{C_1}$ be the payoff of the European basket put at maturity $T$:

$$P_T = \max\{K - BK_T, 0\}.$$ 

Applying the put–call parity, we obtain

$$P_T \approx C_T - BK_T + KP(t, T). \tag{11}$$

The above formula is the closed-form approximation for a vulnerable European basket put warrant.

Next, using the put–call parity we derive the approximate valuation formula for a vulnerable European basket put warrant. Let $P_T^b$ be the payoff of the European basket put at maturity $T$:

$$P_T^b = \max\{K - BK_T, 0\}.$$ 

Applying the put–call parity, we obtain

$$P_T^b \approx C_T^b - BK_T + KP(t, T). \tag{11}$$

It is worth noting that when $n = 1$ the geometric average is equal to the arithmetic average according to equation (8). Substituting $n = 1$ into equation (10), it gives the pricing formula of Klein and Inglis (1999).

In addition, our pricing formula is also the general formula for valuing basket call warrants with stochastic interest rates, not considering credit risk; with fixed interest rate, considering credit risk; with fixed interest rate, considering credit risk; and the unrelated assets.

4. Numerical examples

Unlike Klein and Inglis (1999), who employ Vasicek’s (1977) interest rate model, we use the CIR model with our numerical examples. Both models are one-factor equilibrium models and have the characteristic of mean-reversion. However, Vasicek’s (1977) model can produce negative interest rates when the initial rate starts from a low value, while the CIR model does not. Due to the assumption that the future instantaneous interest rates are normally distributed, we therefore use the CIR model. The dynamics of the short-term interest rates of the CIR model under the risk-neutral measure $Q$ are as follows:

$$dr_t = a(b - r_t) dt + \sigma \sqrt{r_t} dW_t,$$

where $b$ is the long-run mean level of interest rate $r$, $a$ is the parameter that governs the speed of mean-reversion, and $W_t$ is the usual Wiener process. We assume $2ab > \sigma^2$ to ensure that zero interest rates will not occur.

Based on the CIR model, the price at time $t$ of a zero-coupon bond paying one dollar at maturity $T$ is

$$P(t, T) = G(t, T) e^{-H(t, T)r},$$

where

$$G(t, T) = \left(\frac{\phi_1 e^{\phi_2(T-t)}}{\phi_2 (e^{\phi_2(T-t)} - 1) + \phi_1}\right)^{\phi_3},$$

$$H(t, T) = \frac{e^{\phi_1(T-t)} - 1}{\phi_2 (e^{\phi_2(T-t)} - 1) + \phi_1},$$

$$\phi_1 = \sqrt{a^2 + 2\sigma^2}, \quad \phi_2 = \frac{a + \phi_1}{2}, \quad \phi_3 = \frac{2ab}{\sigma^2}.$$ 

The parameters $a$, $b$, and $\sigma$ are all positive. The volatility of the bond return is then

$$\sigma_P(T - t) = H(t, T) \sigma \sqrt{r}.$$